Beijing-Dublin International College (BDIC)

Advanced Mathematics

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Supplemental Reading Book

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Contents

Chapter 1 Set notation and complex numbers

1.1 Imaginary numbers

Suppose that you are asked to solve the equation

$$x^2 + 1 = 0$$

Your first response might be to say that there will be two solutions as it is a quadratic equation. Very quickly you might write down the line

$$x^2 = -1.$$

At that point you might conclude, correctly, that there are no real solutions to the equation. But what if we agree that there exists a number x whose square is -1?

Such a number does indeed exist, although it is not a real number. It is known as an imaginary number. We denote it by i (although some branches of engineering use j instead) and we'll assume that the usual rules for algebraic manipulation apply. Then, since $i^2 = -1$, we also have

$$(-i)^2 = (-1 \times i)^2 = (-1)^2 \times i^2 = 1 \times -1 = -1.$$

The equation $x^2 + 1 = 0$ now has two imaginary solutions, namely i and $-i$.

What about the equations

$$x^2 + 4 = 0 \qquad \text{and} \qquad y^2 + 7 = 0?$$

The first of these has imaginary solutions $x = \pm 2i$, since

$$(\pm 2i)^2 + 4 = 4i^2 + 4 = -4 + 4 = 0.$$

Similarly, the second has solutions $y = \pm i\sqrt{7}$.

Any non–zero real multiple of i is an imaginary number. The square of an imaginary number is a negative real number. For example $3i$, $-20i$, $-i/5$, $0.125i$ and πi are all imaginary numbers, and $(3i)^2 = -9$, $(-20i)^2 = -400$, $(-i/5)^2 = -1/25$ and so on.

1.2 Complex numbers and solutions of quadratic equations

Suppose that you are given this equation to solve:

$$x^2 - 4x + 5 = 0.$$

Completing the square and rearranging gives $(x-2)^2 = -1$; that is, $x - 2 = \pm i$ or $x = 2 \pm i$. These solutions can also be obtained by applying the familiar quadratic formula.

$$x = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4i^2}}{2} = 2 \pm \sqrt{i^2} = 2 \pm i.$$

These solutions are not purely imaginary, although they do involve an imaginary number. The solutions $2+i$ and $2-i$ are complex numbers. They have a real part and an imaginary part. For example, the real part of $2+i$ is 2; we write $\mathrm{R}e(2+i) = 2$. The imaginary part of a complex number is the coefficient of i, so the imaginary part of $2+i$ is 1; we write $\mathrm{I}m(2+i) = 1$. You may like to show by substitution that $2+i$ and $2-i$ are indeed solutions of $x^2 - 4x + 5 = 0$.

✎ ***Example* 1.1.**

i) Consider the complex number $3+8i$. $\mathrm{R}e(3+8i) = 3$ and $\mathrm{I}m(3+8i) = 8$.

ii) If $z = \frac{1}{2} - 5i$ then $\mathrm{R}e(z) = \frac{1}{2}$ and $\mathrm{I}m(z) = -5$.

iii) For the purely imaginary number $-7i$, since $-7i = 0 - 7i$, we have $\mathrm{R}e(-7i) = 0$ and $\mathrm{I}m(-7i) = -7$.

1. For the real number 4 (which can be thought of as the complex number $4+0i$), $\mathrm{R}e(4) = 4$ and $\mathrm{I}m(4) = 0$.

If we allow complex numbers as solutions to quadratic equations with real coefficients then every such quadratic equation will always have solutions, and they will be either both real or both complex.
We can see this in general if we look at the quadratic formula. The solution to the quadratic equation $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Whether $ax^2 + bx + c = 0$ has (purely) real or complex roots depends on the expression $b^2 - 4ac$ which is known as the discriminant of the quadratic.

$$x \text{ is } \begin{cases} \text{real} & \text{if } b^2 - 4ac \geq 0 \\ \text{complex} & \text{if } b^2 - 4ac < 0. \end{cases}$$

✎ ***Example* 1.2.** The solutions of $x^2+6x+25=0$ must be complex since $b^2-4ac=64<0$. Using the quadratic formula, the solutions are found to be $-3+4i$ and $-3-4i$. These complex numbers are related; they are *complex conjugates* of each other. This will be examined further in the next section.

1.3 Arithmetic with complex numbers

Complex numbers can be added or multiplied together, subtracted from one another or divided by one another.

Consider two complex numbers $z=a+bi$ and $w=c+di$. Here the real part of z is a and the imaginary part of z is b; the real part of w is c and the imaginary part of w is d.

Addition

$$\begin{aligned} z+w &= (a+bi)+(c+di) \\ &= (a+c)+(b+d)i \end{aligned}$$

Rule: Add real parts to real parts and imaginary parts to imaginary parts.

✎ ***Example* 1.3.**

$$\begin{aligned} (3-4i)+(1+2i) &= 3+1+(-4+2)i \\ &= 4-2i \end{aligned}$$

♢

Substraction

$$\begin{aligned} z-w &= (a+bi)-(c+di) \\ &= (a-c)+(b-d)i \end{aligned}$$

Rule: Subtract real parts from real parts and imaginary parts from imaginary parts.

✎ ***Example* 1.4.**

$$\begin{aligned} (3-4i)-(1+2i) &= 3-1+(-4-2)i \\ &= 2-6i \end{aligned}$$

♢

Multiplication

$$\begin{aligned} zw &= (a+bi)(c+di)=ac+adi+bci+(bd)i^2 \\ &= (ac-bd)i+(ad+cb)i \end{aligned}$$

Rule: Expand the brackets in the normal way, remembering that i^2 can be simplified to -1, and collect terms into real and imaginary parts.

✎ ***Example* 1.5.**

$$(3-4i)(1+2i) = 3-4i+6i-8i^2 = 3+2i+8 = 11+2i$$

$\Diamond$

To divide one complex number by another we need to know about the complex conjugate.

***Definition* 1.1.** The complex conjugate of the complex number $z = a+ib$ is the complex number denoted by $\overline{z}$, where $\overline{z} = a - ib$.

Notice that if $z = a+ib$ then $z\overline{z} = \overline{z}z = a^2+b^2$. In particular, $z\overline{z}$ is always a non-negative real number and $z\overline{z} = 0$ if and only if $z = 0$. This observation is exactly what we need when dividing one complex number by another (non–zero) complex number.

✎ ***Example* 1.6.**

i) $\overline{3+5i} = 3-5i$

ii) $\overline{2-7i} = 2+7i$

iii) If z is a real number then $\overline{z} = z$. If z is a purely imaginary number then $\overline{z} = -z$. For example $\overline{3i} = -3i$.

$\Diamond$

Division

If $w \neq 0$ then to find $\frac{z}{w}$ we multiply both top and bottom by the complex conjugate of w.

$$\begin{aligned} \frac{z}{w} &= \frac{z}{w}\frac{\overline{z}}{\overline{w}} = \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)} = \frac{ac-adi+cbi-(bd)i^2}{c^2-cdi+cdi-d^2i^2} \\ &= \frac{(ac+bd)+(cb-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{cb-ad}{c^2+d^2}i \end{aligned}$$

This process is similar to rationalizing the denominator of a quotient of surds. Multiplying by the complex conjugate of the divisor produces a real number in the denominator and allows the number to be written in the form $a+bi$.

✎ ***Example* 1.7.**

$$\begin{aligned} \frac{5-10i}{1+2i} &= \frac{(5-10i)(1-2i)}{(1+2i)(1-2i)} = \frac{5-20i+20i^2}{1-2i+2i-4i^2} \\ &= \frac{-15-20i}{5} = -3-4i \end{aligned}$$

$\Diamond$

Equality

Two complex numbers are equal to each other if and only if both their real and imaginary parts are equal. In other words, if $z = a+bi$ and $w = c+di$, then $z = w$ if and only if $a = c$ and $b = d$.

1.4 The set of complex numbers

We can think of a real number as a particular type of complex number, one with zero imaginary part. The complex numbers include real numbers and form a set which contains the set of real numbers and hence all of the other number sets we have mentioned.

> ***Definition* 1.2.** The set of complex numbes $\mathbb{C}$ is the set of all numbers of the form $a + ib$ where a and b are real numbers and $i^2 = -1$.

Altematively, we could write

$$\mathbb{C} = \{a + ib | a, b \in \mathbb{R}, i^2 = -1\}.$$

We also have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

The set of complex numbers, like the set of real numbers, is closed under addition, subtraction, multiplication and division.

Complex numbers, however, lack an important property of the real numbers.

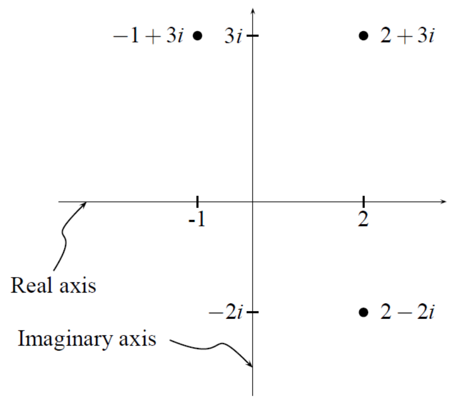
The set of real numbers is ordered; that is, if we have any two real numbers x and y we can say that either $x > y$ or $x < y$ or $x = y$. One of these alternatives will always be true. Because of this property we are able to represent real numbers on the real number line.

The set of complex numbers is not ordered. Consider the two complex numbers $2 - 3i$ and $-1 + 5i$. Clearly $2 - 3i \neq -1 + 5i$ as neither their real nor their imaginary parts are the same. But it makes no sense to write $2-3i > -1+5i$ or $2-3i < -1+5i$. It does make sense to write $\mathrm{R}e(2-3i) > \mathrm{R}e(-1+5i)$ and $\mathrm{I}m(2-3i) < \mathrm{I}m(-1+5i)$ but this is because the real part and the imaginary part of a complex numbers are both real numbers.

Because the set of complex numbers is not ordered, complex numbers cannot be represented as points on a line. Instead, complex numbers are represented as points on a plane.

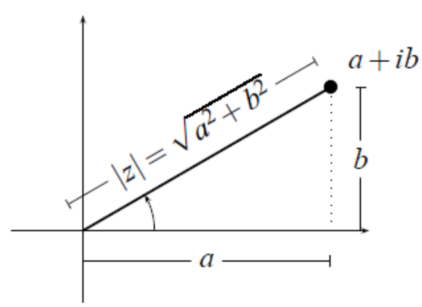
The complex plane

The *complex plane* or *Argand diagram* allows complex numbers to be represented graphically. The horizontal axis in the complex plane is called the *real axis*. All real numbers lie on the horizontal axis in the complex plane; positive numbers to the right of the origin, negative numbers to its left. The vertical axis is known as the *imaginary axis*. All purely imaginary numbers lie on the vertical axis. Each point in the complex plane corresponds to a single complex number. This is a little different to the Cartesian plane used in coordinate geometry, where each point corresponds to an ordered pair of real numbers. For example:



1.4.1 The modulus of a complex number

For a real number x, the modulus of x, written as $|x|$, gives the distance on the real number line from x to the origin (zero). For a complex number $z = a + bi$, the modulus of z, written $|z|$, gives the distance in the complex plane from z to the origin.



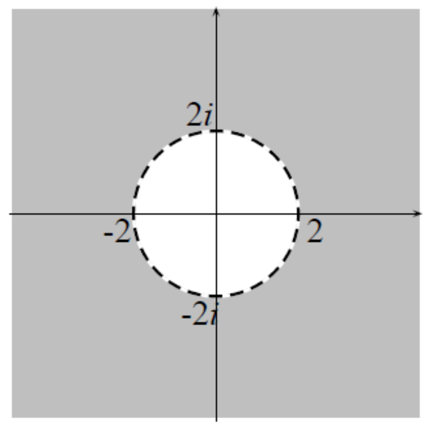
If $z = a + bi$ the geometrically, by Pythagoras' Theorem

$$|z| = \sqrt{a^2 + b^2}$$

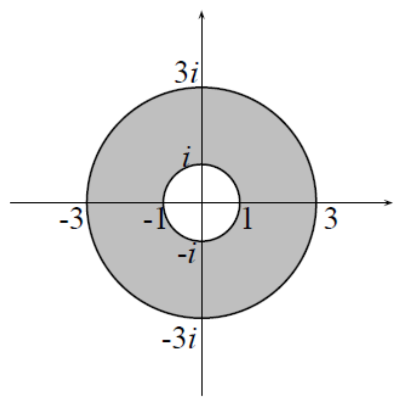
(This formula holds in all quadrants of the complex plane.) Alternatively we can express $|z|$ in terms of z and its complex conjugate. Since $z\overline{z} = (a + bi)(a - bi) = a^2 + b^2$ we can write $|z| = \sqrt{z\overline{z}}$.
The modulus of a complex number is a real number and so it makes sense to write something like $|1 + i| < |2 - 3i|$; however two complex numbers with the same modulus need not be equal. For example $|4 - 3i| = |1 + 2\sqrt{6}i| = 5$. Note that the modulus is always a positive real number or zero.
The modulus can be used to specify subsets of the set of complex numbers which can be graphed in the complex plane.

✎ ***Example* 1.8.**

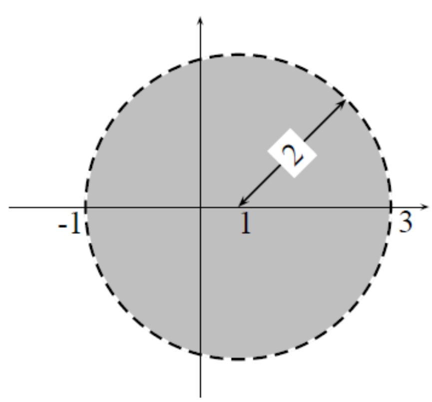
i) $\{z \in \mathbb{C} \mid |z| > 2\}$ is the set of complex numbers z such that z is more than 2 units distant from the origin. The set is represented by the shaded area below, which extends indefinitely.



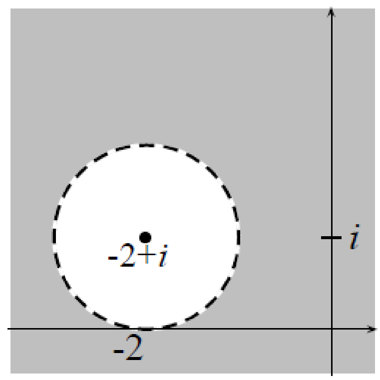
ii) $\{z \in \mathbb{C} | 1 \leq |z| \leq 3\}$ is the set of complex numbers which are between one and three units distant from the origin.



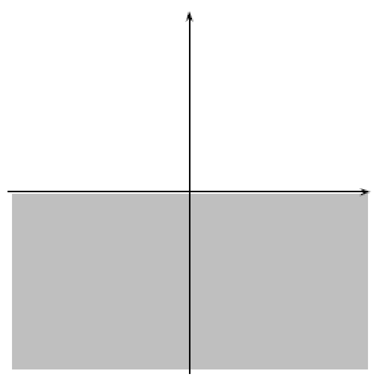
iii) $\{z \in \mathbb{C} || z-1| < 2\}$. As with real numbers, $|z-1|$ is exactly the distance from z to 1. Hence, this is the set of all complex numbers whose distance from 1 is less than 2. Geometrically, these are all points in the complex plane that are inside the circle, centre 1, radius 2.



iv) $\{z \in \mathbb{C} || z+2-i| > 1\}$. Here $|z+2-i| = |z-(-2+i)|$ is the distance from the complex number z to $-2+i$. So this set is the set of all complex numbers whose distance from $-2+i$ is greater than one unit. In other words, this is the set of points in the complex plane with are strictly outside the circle of radius 1 and centre $-2+i$.



v) Here is a different type of subset of the complex numbers: $z \in \mathbb{C} | \mathrm{I}m\, z \leq 0$ is the set of all complex number whose imaginary part is less or equal to zero.



$\Diamond$

1.4.2 Some properties of the modulus

For all complex numbers $z = a+ib$ and $w = c+id$, we have

1. $|zw| = |z||w|$,

2. $|z/w|=|z|/|w|$,

3. $|z+w|\leq |z|+|w|$,

4. $|z-w|\geq |z|-|w|$.

To show that (i) is true, we calculate $|zw|$ and $|z||w|$ separately and show they are equal.

$$\begin{aligned}|zw| &= |(a+ib)(c+id)| = |(ac-bd)+i(ad+bc)| \\ &= \sqrt{(ac-bd)^2+(ad+bc)^2} = \sqrt{a^2c^2+b^2d^2-2abcd+a^2d^2+b^2c^2+2abcd} \\ &= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}\end{aligned}$$

while

$$\begin{aligned}|z|\,|w| &= \sqrt{a^2+b^2}\sqrt{c^2+d^2} \\ &= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} = |zw|\end{aligned}$$

We will show (iii) algebraically (a geometric argument can also be used, and this is left as an exercise). Note that since all quantities are non–negative, proving $|z+w|\leq |z|+|w|$ is equivalent to proving $|z+w|^2\leq (|z|+|w|)^2$. Now

$$|z+w|^2 = |(a+c)+i(b+d)|^2 = (a+c)^2+(b+d)^2 = a^2+b^2+c^2+d^2+2(ac+bd).$$

Similarly,

$$(|z|+|w|)^2 = |z|^2+|w|^2+2|z|\,|w| = a^2+b^2+c^2+d^2+2|z|\,|w|\,.$$

So we must prove that $ac+bd\leq |z||w|$. Now

$$\begin{aligned}|z|\,|w| &= \sqrt{a^2+b^2}\sqrt{c^2+d^2} = \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} = \sqrt{(ac+bd)^2+(ad-bc)^2} \\ &\geq \sqrt{(ac+bd)^2} = |ac+bd|\end{aligned}$$

Now every real number k is less than or equal to its own absolute value $|k|$. Hence

$$ac+bd\leq |ac+bd|\leq |z||w|,$$

and the proof is complete.

The proofs of (ii) and (iv) are left as an exercise.

In the next chapter we will explore some uses of the complex plane representation of complex numbers and show how an understanding of the geometry of complex numbers is useful in performing certain types of calculations.

✍ **Exercises** In addition to doing the following exercises you should look at the online quiz `www.maths.usyd.edu.au/u/UG/JM/MATH1001/Quizzes/quiz1.html.` which covers the material in this chapter. You can get to this page from the course homepage.

1. In each of the following exercises, perform the indicated operations and give the final answer in the form $x + iy$.

   a) $(5 - 2i) + (2 + 3i)$
   b) $(2 - i) - (6 - 3i)$
   c) $(2 + 3i)(-2 - 3i)$
   d) $-i(5 + i)$
   e) $1/i$
   f) $(a + ib)(a - ib)$
   g) $6i/(6 - 5i)$
   h) $(a + ib)/(a - ib)$
   i) $1/(3 + 2i)$
   j) $i^2, i^3, i^4, ..., i^1 0$
   k) $(1 + i)/(1 - i)$
   l) $[i/(1 - i)] + [(1 - i)/i]$
   m) $(1/i) - 3i(1 - i)$
   n) $i^1 23 - 4i^9 - 4i$

2. If $z = 5 + 12i$ and $w = 3 + 4i$, express $w + z$, $z - w$, zw and z/w in the form $a + ib$. Use these results to verify that

   a) $|zw| = |z||w|$
   b) $|z/w| = |z|/|w|$
   c) $|z + w| \leq |z| + |w|$
   d) $|z - w| \geq |z| - |w|$

3. If $z = x + iy$, express each of the following explicitly in terms of x and y.

   a) $\mathrm{Re}(z/\overline{z})$
   b) $|(z/\overline{z})|$
   c) $\mathrm{Im}\, z^3$
   d) $\mathrm{Re}\, z^4$
   e) $|z^6|$
   f) $|(z + 1)/(z - 1)|$
   g) $\mathrm{Re}(1/z^2)$

4. Simplify the following expressions.

   a) $\mathrm{Im}\, \frac{1}{1+i}$
   b) $\mathrm{Re}\, \frac{(1-i)^2}{1+2i}$
   c) $|\cos\theta + i\sin\theta|$, where θ is any angle
   d) $|\frac{1+3i}{3+i}|$
   e) $|\frac{(1+i)^6}{i^3(1+4i)^2}|$

5. Solve the following equations using the quadratic formula.

   a) $y^2 + 2y + 5 = 0$
   b) $z^2 + 3z + 8 = 0$
   c) $t^2 + t - 1 = 0$
   d) $7a^2 + 8a + 4 = 0$

6. If $z = 3 - 2i$, plot z, $-z$, $\overline{z}$ and $-z$ as points in the complex plane.

7. Show that for any complex number z, $|\overline{z}| = |z|$.

8. $\overline{z} = z$, what can you say about z?

9. Prove properties (ii) and (iv) of the modulus, given at the end of the chapter.
(Hint for (iv): write $|z| = |(z+w)w|$ and use property (iii).)

10. Give a geometric justification of the *triangle inequality*:

$$|z_1 + z_2| \leq |z_1| + |z_2|,$$

where z_1 and z_2 are any two complex numbers.

11. In each of the following cases, find the set of all points in the complex plane satisfying the given condition (describe the set, sketch it, and give its cartesian equation, if appropriate).

a)	$\mathrm{I}m\, z \geq 0$	f)	$\lvert z-5 \rvert = 6$
b)	$0 < \mathrm{I}m(z+1) \leq 2\pi$	g)	$\lvert z+2i \rvert \geq 1$
c)	$-1 \leq \mathrm{R}e\, z < 1$	h)	$\lvert z+i \rvert = \lvert z-i \rvert$
d)	$\mathrm{R}e(iz) = 3$	i)	$\lvert z+3 \rvert + \lvert z+1 \rvert = 4$
e)	$\mathrm{R}e(z+2) = -1$	j)	$\lvert z+3 \rvert - \lvert z+1 \rvert = \pm 1$

12. If z is a variable complex number, mark clearly on an Argand diagram (i.e., on the complex plane) the regions described by

a)	$\mathrm{R}e\, z \geq 2$ and $0 \leq \mathrm{I}m(z) \leq 3$	d)	$\lvert z-2+i \rvert > 6$ and $\mathrm{R}e\, z > 2$
b)	$\mathrm{R}e\, z \geq -2$ or $0 \leq \mathrm{I}m\, z \leq 3$	e)	$1 < \lvert z-2+i \rvert < 3$ and $\mathrm{I}m\, z \geq 0.$
c)	$2 < \lvert z \rvert < 3$ and $\mathrm{R}e\, z < 2.5$		

◊

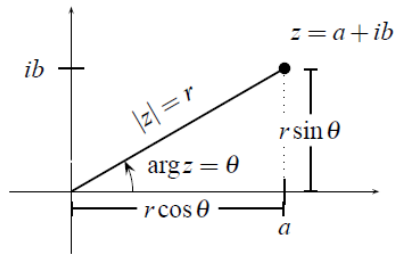
Chapter 2 Polar form and roots of complex numbers

A detailed list of mathematical objectives (knowledge, understanding and skills) for this chapter is provided in the weekly tutorial exercises. Equally important are *the generic skills you will develop*. The study of mathematics enhances your ability to think logically and analytically, move from the particular to *the general, work quantitatively and improve problem-solving skills*. By reading and working carefully through the material in this chapter you will develop the following additional generic skills:

- Recognise that objects can be represented in different but equivalent forms.
- Develop the ability to choose the form most suitable for the task at hand.
- Use geometric/visual techniques to help understand new concepts.

2.1 Polar and Cartesian forms of complex numbers

In the last chapter we introduced the set of complex numbers and showed how such numbers can be represented as points on the complex plane. To position the complex number$z = a + bi$ we need two pieces of information: the real part a and the imaginary part b. We can also plot the same number, however, with two different pieces of information: the distance $|z|$ of the point from the origin and the angle of the line from the point to the origin, measured anticlockwise from the positive real axis. This angle is known as the argument of z or arg z.



If we let $|z|= r$ and $\arg z = \theta$ then we can see that

$$\mathrm{Re}\, z = a = r\cos\theta \text{ and } \mathrm{Im}\, z = b = r\sin\theta.$$

Therefore instead of writing $z = a + ib$, we can write

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

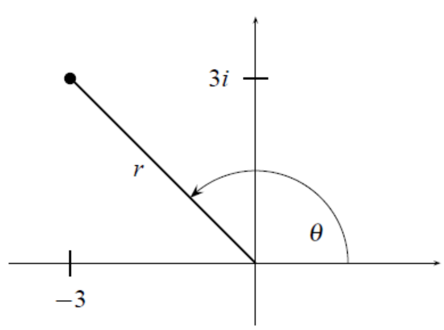
This is known as the *polar form* of a complex number. In polar form, a complex number is specified by its modulus r and its argument θ . The form $z = a + ib$ introduced in the last chapter is called the *Cartesian form*, in which z is specified by its real and imaginary parts.

Complex numbers can easily be changed from one form to another. If a number is in Cartesian form $z = a + ib$, then the modulus r is equal to $\sqrt{a^2 + b^2}$ and the argument θ can be found using $\tan\theta = \frac{b}{a}$. Because $\tan\theta$ has the same values in the first and third quadrants and in the second and fourth quadrants *it is essential that you plot the complex number on the complex plane when you are finding its argument*. This will make it very clear in which quadrant the argument lies.

✎ ***Example* 2.1.**

1. Write $-3 + 3i$ in polar form.

   Here $r = |-3 + 3i|= \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$. Plotting $-3 + 3i$ on the complex plane gives the following picture.

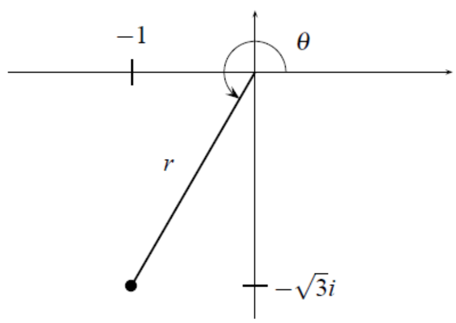


Hence arg$(-3+3i)$ is an angle in the second quadrant. By inspection we can see that arg$(-3+3i) = \frac{3\pi}{4}$. Alternatively we find that$\tan\theta = -1$ and hence $\theta = 3\pi/4$. Without the diagram we are left with the alternatives $\theta = -\pi/4$ or $3\pi/4$. As we know, $\tan^{-1}(-1) = -\pi/4$. (If you use a calculator in radian mode, it will tell you that $\tan^{-1}(1) \approx -0.7854$.) The diagram easily distinguishes between right and wrong answers. So

$$-3+3i = 3\sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4).$$

2. Write $-1-\sqrt{3}i$ in polar form.

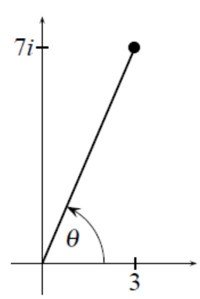
   The modulus is given by $r = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$. We plot $-1-\sqrt{3}i$ in the complex plane.



   We see that arg$(-1+\sqrt{3}i)$ lies in the third quadrant. Since $\tan\theta = \sqrt{3}$ the value of θ is $4\pi/3$. (We could also write $\theta = -2\pi/3$ equally correctly.) Therefore $-1-\sqrt{3}i = 2(\cos 4\pi/3 + i\sin 4\pi/3)$ in polar form.

3. Find the modulus and argument of $3+7i$.

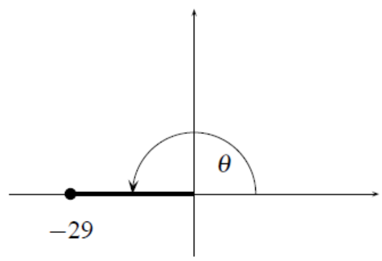
   The modulus is $r = \sqrt{3^2+7^2} = \sqrt{58}$. In the complex plane $3+7i$ lies in the first quadrant.



We find that $\tan\theta = \frac{7}{3}$ adn so $\theta^{-1}\frac{7}{3} \approx 1.17$. In polar form $3 + 7i = \sqrt{58}(\cos(\tan^{-1}\frac{7}{3}) + i\sin(\tan^{-1}\frac{7}{3})) \approx \sqrt{58}(\cos 1.17 + i\sin 1.17)$

4. Write -29 in polar form.

   Although -29 is a real number it can still be written in polar form. Clearly $|-29| = 29$ and from the complex plane we see $\arg(-29) = \pi$.

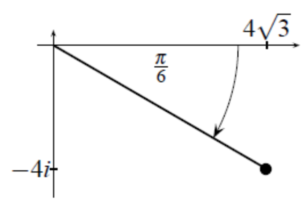


   Hence $-29 = 29(\cos\pi + i\sin\pi)$ in polar form.

5. Convert $8(\cos(-\pi/6) + i\sin(-\pi/6))$ to Cartesian form.

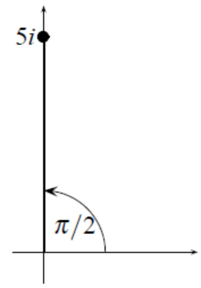
   It is usually much simpler to convert a complex number from polar form to Cartesian form than to convert a complex number from Cartesian to polar form. All that needs to be done is to evaluate the cosine and sine and simplify the resulting expression. So

$$8(\cos(-\pi/6) + i\sin(-\pi/6)) = 8\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 4\sqrt{3} - 4i$$



6. Convert $5(\cos(\pi/2) + i\sin(\pi/2))$ into Cartesian form.

$$5(\cos(\pi/2) + i\sin(\pi/2)) = 5\,(0 + i) = 5i$$



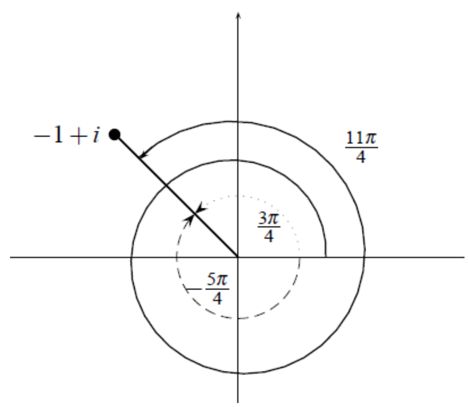
◊

Sometimes the polar form of a complex number, $r(\cos\theta + i\sin\pi)$ is abbreviated to r cisπ, where

$$\text{cis}\theta = \cos\theta + i\sin\theta$$

So, for example, $8\text{cis}(\frac{-\pi}{6}) = 8(\cos(\frac{-\pi}{6}) + i\sin(\frac{-\pi}{6}))$

For a complex number z we can choose how to express arg$z$. For example, arg$(-1+i)$ can be given as $3\pi/4$ or $-5\pi/4$ or $11\pi/4$. We could write it most generally as $\frac{3\pi}{4} + 2k\pi$ where $k \in \mathbb{Z}$. In fact, a given complex number has an infinite number of arguments which differ by integer multiples of 2π. This fact becomes important when we take roots of complex numbers later in this chapter.



To eliminate this ambiguity we can specify the principal argument of z, Arg z:

> The principal argument of z, Arg$z$, is the particular argument of z such that
>
> $$-\pi < \text{Arg}z \leq \pi$$

Hence arg$(-1+i) = 3\pi/4$ or $-5\pi/4$ or $11\pi/4$ and so on, but Arg$z = 3\pi/4$ only.

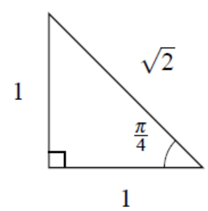
It is important to understand that complex numbers have multiple arguments when equating two complex numbers in polar form. Consider the complex numbers $z = r(\cos\theta + i\sin\theta)$ and $w =$

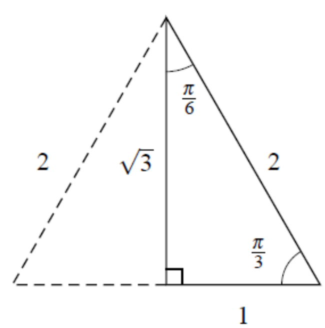
$s(\cos\phi + i\sin\phi)$. If$z = w$ then they both correspond to the same point in the complex plane. Hence they are the same distance from the origin (that is, $r = s$) and have the same principal argument (that is, $\theta = \phi + 2k\pi$, where k is an integer).

☞ **[Aside]:**

A note on special angles. In the examples above you will see that most of the polar angles that we used were angles with exact sines or cosines, sometimes known as special angles. For example, any angle which is a multiple of p/2 has either sine or cosine equal to zero. So $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$ and $\cos\pi = -1$, $\sin\pi = 0$.

The angles $\pi/6$ and $\pi/3$, which correspond to 30 and 60 degrees respectively (and any angles that are multiples of these) have special values for sine and cosine, as does $\pi/4$ (45 degrees) and its multiples. You will have learnt about these special cases at high school. As they are used extensively in this chapter, it is important that you revise them as soon as possible if you have forgotten about them. You may find it helpful to look at the right-angle triangles with angle $\pi/4$ or $\pi/3$ and $\pi/6$.





It is easy to find sines and cosines from these triangles, since $\sin\theta$ is given by the length of the side opposite θ divided by the length of the hypotenuse and $\cos\theta$ is given by the length of the side adjacent to θ divided by the length of the hypotenuse. For example, $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}2$ and $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$. ✌

2.2 Arithmetic in polar form

Complex numbers in polar form can be added or multiplied together, subtracted from one another, divided by one another or raised to a power. We shall see, however, that polar form is particularly useful when multiplying or dividing complex numbers, or raising a complex number to a power.
Consider two complex numbers: $z = r(\cos\theta + i\sin\theta)$ and $w = t(\cos\phi + i\sin\phi)$. Here the modulus of z is r and θ is an argument of z. The modulus of w is t and ϕ is an argument of w.

Addition and subtraction

$$\begin{aligned} z + w &= r(\cos\theta + i\sin\theta) + t(\cos\phi + i\sin\phi) \\ &= r\cos\theta + ir\sin\theta + t\cos\phi + it\sin\phi \\ &= (r\sin\theta + t\cos\phi) + i(r\sin\theta + t\sin\phi) \end{aligned}$$

Subtraction is done in a similar way. Generally, there is little point in changing a complex number from Cartesian form to polar form to perform addition or subtraction. Using polar form for addition and subtraction is more complicated and gives no extra insight to the problem.

✎ ***Example* 2.2.**

$$\begin{aligned} 6(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) - 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) &= 6\cos\frac{\pi}{3} + 6i\sin\frac{\pi}{3} - 2\cos\frac{\pi}{3} - 2i\sin\frac{\pi}{3} \\ &= (6\cos\frac{\pi}{3} - 2\cos\frac{\pi}{6}) + i(6\sin\frac{\pi}{3} - 2\sin\frac{\pi}{6}) \end{aligned}$$

♢

The solution in this example, although it involves sines and cosines, is no longer in polar form; polar form is *strictly* an expression of the type $r(\cos\theta + i\sin\theta)$.

Multiplication

$$\begin{aligned} zw &= r(\cos\theta + i\sin\theta)t(\cos\phi + i\sin\phi) \\ &= rt(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rt(\cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi + i^2\sin\theta\sin\phi) \\ &= rt((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\ &= rt(\cos(\theta + \phi) + i\sin(\theta + \phi)). \end{aligned}$$

The last line uses the angle sum formulas of trigonometry:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta. \end{aligned}$$

Notice that multiplication of two numbers in polar form gives an *answer in polar form*. We see that the modulus of the product is rt, the product of the two moduli of the numbers which we multiplied together, and that $(\theta + \phi)$, the sum of the arguments of the original two numbers, is an argument of the product. In general:

To multiply complex numbers in polar form *multiply* the moduli and *add* the arguments. That is,

$$r(\cos\theta + i\sin\theta) \times t(\cos\phi + i\sin\phi) = rt(\cos(\theta+\phi) + i\sin(\theta+\phi))$$

✎ ***Example* 2.3.**

$$6(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) \times 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 12(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$$

We can write down the answer straight away as the modulus is the product of 6 and 2 and $\pi/3 + \pi/6 = \pi/2$ is an argument. $\Diamond$

Division

$$\begin{aligned}\frac{z}{w} &= \frac{r(\cos\theta + i\sin\theta)}{t(\cos\phi + i\sin\phi)} \\ &= \frac{r(\cos\theta + i\sin\theta)t(\cos\phi - i\sin\phi)}{t(\cos\phi + i\sin\phi)t(\cos\phi - i\sin\phi)}\end{aligned}$$

You should note that a complex number in the form $t(\cos\phi - i\sin\phi)$ is *not* in polar form and so the rule for multiplication in polar form does not apply. However, a number of the form $t(\cos\phi - i\sin\phi)$ can easily be put into polar form because

$$-\sin\phi = \sin(-\phi) \text{ and } \cos\phi = \cos(-\phi)$$

Consequently,

$$t(\cos\phi - i\sin\phi) = t(\cos(-\phi) + i\sin(-\phi)),$$

and hence,

$$\begin{aligned}\frac{z}{w} &= \frac{r\,(\cos\theta + \sin\theta)\,t\,(\cos(-\phi) + i\sin(-\phi))}{t(\cos\phi + i\sin\phi)t(\cos\phi - i\sin\phi)} \\ &= \frac{rt(\cos\theta + i\sin\theta)(\cos(-\phi) + i\sin(-\phi))}{t^2(\cos^2\phi + \sin^2\phi)} \\ &= \frac{r}{t}\left(\cos(\theta - \phi) + i\sin(\theta - \phi)\right).\end{aligned}$$

We see that the modulus of the quotient is r/t, the quotient of the two moduli of the original two numbers, and that $(\theta - \phi)$, the difference of the arguments of the original two numbers, is an argument of the quotient. In general:

> To divide complex numbers in polar form we *divide* their moduli and *subtract* their arguments.

✎ ***Example* 2.4.**

$$\frac{6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)} = \frac{6}{2}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right) = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

We can write down the answer immediately as the modulus is the quotient of 6 and 2 and an argument is $\pi/3 - \pi/6 = \pi/6$. ♢

Raising to an integer power Let us consider the problem $(r(\cos\theta + i\sin\theta))^2$. Using the rule for multiplication in polar form this becomes $r^2(\cos 2\theta + i\sin 2\theta)$. Following on from this we can write

$$(r(\cos\theta + i\sin\theta))^3 = r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta).$$

Similarly $(r(\cos\theta + i\sin\theta))^4 = r^4(\cos 4\theta + i\sin 4\theta)$. It is easy to see that for $n \in \{1, 2, 3, ...\}$.

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta).$$

It is a useful exercise to prove this by induction. In fact, this is true not only when n is a positive integer but for all integer values of n. Note that $(r(\cos\theta + i\sin\theta))^0 = 1$.

> To raise a complex number to any integer, raise the modulus to the integer and multiply the argument by the integer.

✎ ***Example* 2.5.**

$$\left(6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)^8 = 6^8\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right) = 6^8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

since $\cos\frac{8\pi}{3} = \cos\frac{8\pi}{3}$ and $= \sin\frac{8\pi}{3} = \sin\frac{2\pi}{3}$. ♢

In the special case when a complex number of modulus 1 is raised to an integer power, we have De Moivres theorem :

> ***Theorem* 2.1.** For any $n \in \mathbb{Z}$,
>
> $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

✎ ***Example* 2.6.**

i)

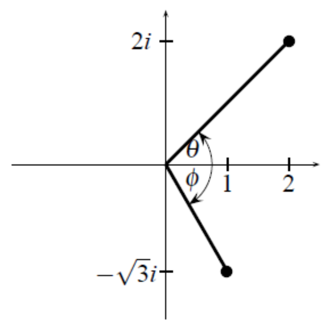
$$3\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\left(4\left(\cos\frac{-\pi}{2}+i\sin\frac{-\pi}{2}\right)\right)=12\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

ii)

$$\frac{3\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)}{4\left(\cos\frac{-\pi}{2}+i\sin\frac{-\pi}{2}\right)}=\frac{3}{4}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right).$$

iii) Find $(2+2i)(1\sqrt{3}i)$ in polar form.

First, let us put both numbers into polar form. This simplifies the multiplication and we will also need these numbers in polar form for the next example. It is essential to draw a diagram:



Here $|2+2i|=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$ and $|1-\sqrt{3}i|=\sqrt{1+3}=2$. From the diagram, $\theta=\arg(2+2i)$ is in the first quadrant and $\phi=\arg(1-\sqrt{3}i)$ is in the fourth quadrant. Since $\tan\theta=1$, $\theta=\pi/4$ and since $\tan\phi=\sqrt{3}$, $\phi=-\pi/3$. So we have

$$\begin{aligned}(2+2i)\left(1-\sqrt{3}i\right)&=2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)2\left(\cos\frac{-\pi}{3}+i\sin\frac{-\pi}{3}\right)\\&=4\sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{-\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{-\pi}{3}\right)\right)\\&=4\sqrt{2}\left(\cos\frac{-\pi}{12}+i\sin\frac{-\pi}{12}\right).\end{aligned}$$

iv) Find $(2+2i)/(1-\sqrt{3}i)$ in polar form.

The numbers $(2+2i)$ and $(1-\sqrt{3}i)$ are already in polar form from the previous example.

$$\begin{aligned}\frac{2+2i}{1-\sqrt{3}}i &= \frac{2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{2\left(\cos\frac{-\pi}{3}+i\sin\frac{-\pi}{3}\right)}\\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}-\frac{-\pi}{3}\right)+i\sin\left(\frac{\pi}{4}-\frac{-\pi}{3}\right)\right)\\ &= \sqrt{2}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right).\end{aligned}$$

v) Find $((2+2i)/(1-\sqrt{3}i))^6$.

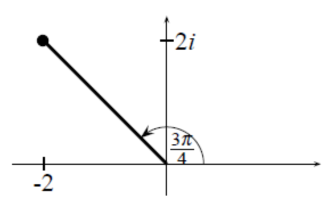
The quotient has already been calculated in polar form in the previous example.

$$\begin{aligned}&\left(\frac{2+2i}{1-\sqrt{3}i}\right)^6\\ &= \left(\sqrt{2}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)\right)^6 = \left(2^{\frac{1}{2}}\right)^6\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)\\ &= 2^3\left(\cos\frac{-\pi}{2}+i\sin\frac{-\pi}{2}\right) = -8i.\end{aligned}$$

◊

2.3 Roots of complex numbers

What is meant by "a root of a complex number", and how could such numbers be found? For example, what is a cube root of $-2+2i$? By analogy with roots of real numbers, an obvious answer is to say its a complex number z whose cube is $-2+2i$. It turns out that the easiest way to find z is to use polar form; if we let $z = r(\cos\theta + i\sin\theta)$ then our task is to find all values of r and all values of θ such that $r^3(\cos 3\theta + i\sin 3\theta) = -2+2i$. Lets also put $-2+2i$ into polar form.



The modulus is $|-2+2i| = \sqrt{8}$. The diagram shows that the principal argument is $3\pi/4$ (in the second quadrant) and so $-2+2i = \sqrt{8}(\cos 3\pi/4 + i\sin 3\pi/4)$. Then we have

$$r^3(\cos 3\theta + i\sin 3\theta) = \sqrt{8}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}).$$

Since the left and right hand sides are both represented by the same point in the complex plane, we must have $r^3 = \sqrt{8}$ and $3\theta = 3\pi/4 + 2k\pi$ where k can be any integer. Therefore

$$r = \sqrt{2} \text{ and } \theta = \pi/4 + 2k\pi/3.$$

The unknown z we seek is then given in its most general form as

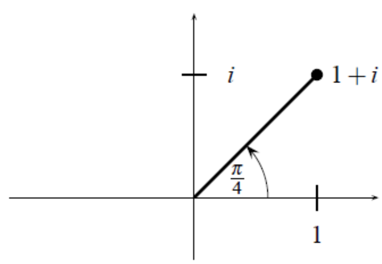
$$z = \sqrt{2}(\cos(\pi/4 + 2k\pi/3) + i\sin(\pi/4 + 2k\pi/3).$$

Since k can be any integer it appears at first sight that there are infinitely many complex numbers z whose cube is $-2+2i$. In fact, it turns out that there are exactly three. To see this, let's experiment with some different values of the integer k.

When $k = 0$, we obtain

$$\begin{aligned} z &= \sqrt{2}(\cos\pi/4 + i\sin\pi/4) \\ &\sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) \\ &+ 1 + i. \end{aligned}$$

Plotting this answer on the complex plane we get:



When $k = 1$, we obtain

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)\right) = \sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right).$$

When $k = 2$, we obtain

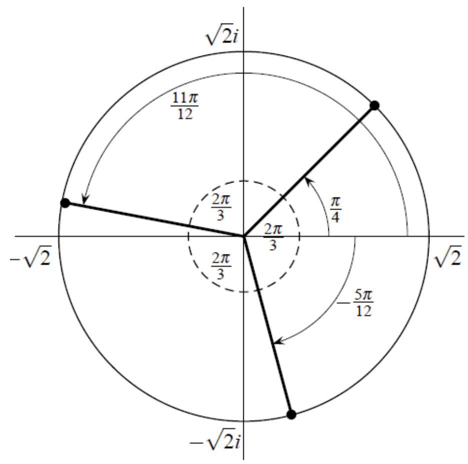
$$\begin{aligned} z &= \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{4\pi}{3}\right)\right) \\ &= \sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) = \sqrt{2}\left(\cos\frac{-5\pi}{12} + i\sin\frac{-5\pi}{12}\right). \end{aligned}$$

If we choose other values of k it turns out that we simply replicate one of the three values of z that weve already calculated. For example, if $k = -1$, then

$$\begin{aligned} z &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{-2\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{-2\pi}{3}\right)\right) \\ &= \sqrt{2}\left(\cos\frac{-5\pi}{12}+i\sin\frac{-5\pi}{12}\right). \end{aligned}$$

which is one of the values already found.

If all three distinct solutions are plotted on the complex plane we see that all lie on the circle of radius $\sqrt{2}$ centred on the origin, and each is separated from the others by an angle of $2\pi/3$.



How many complex roots does a real number have? Let us look at the fourth roots of 16. You already know that $2^4 = (-2)^4 = 16$. Hence 2 and -2 are fourth roots of 16. Are there other fourth roots?

We are looking for all z such that $z^4 = 16$. Writing $z = r(\cos\theta + i\sin\theta)$ and $16 = 16(\cos 0 + i\sin 0)$ we have

$$z^4 = r^4(c\cos 4\theta + i\sin 4\theta) = 16(\cos 0 + i\sin 0)$$

and hence $r^4 = 16$ and $4\theta = 0 + 2k\pi = 2k\pi$, where k can be any integer. Therefore $r = 2$ and $\theta = k\pi/2$, for any integer k. This gives $z = 2(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2})$.

We shall now choose various values of k to find explicit values of z.

When $k = 0$ we obtain

$$z = 2(\cos 0 + i\sin 0) = 2.$$

When $k = 1$ we obtain

$$z = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2i.$$

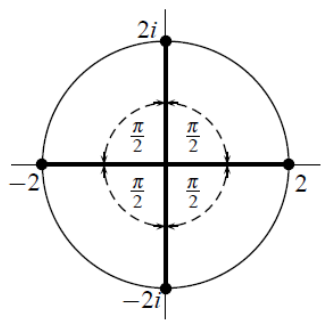
When $k = 2$ we obtain

$$z = 2(\cos\pi + i\sin\pi) = -2,$$

and when $k = 3$ we obtain

$$z = (\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = -2i.$$

All other values of k give one of the four answers already found, namely ± 2, $\pm 2i$. For example, if $k = 7$, we obtain $z = 2(\cos\frac{7\pi}{2} + i\sin\frac{7\pi}{2}) = -2i$.
Therefore, 16 has two real fourth roots (± 2) but it has four complex fourth roots $(\pm 2, \pm 2i)$. When these roots are plotted on the complex plane, they all lie on the circle of radius 2 centred at 0, spaced $\pi/2$ apart.

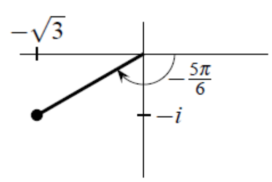


In fact, every non–zero complex number (which includes every real number) has two complex square roots, three complex cube roots, four complex fourth roots and so on. In general:

Every non–zero complex number has n distinct complex nth roots.

Therefore if we seek to find all cube roots of a complex number, for example, we know that there will be three of them. Knowing how many roots to look for is useful in deciding which different values of k to use in finding the roots explicitly.

✎ ***Example* 2.7.** Find all fifth roots of $-\sqrt{3} - i$, that is, all z such that $z^5 = -\sqrt{3} - i$.
First, we put $-\sqrt{3} - i$ into polar form. The modulus of $-\sqrt{3} - i$ is $|-\sqrt{3} - i| = 2$. Plotting $-\sqrt{3} - i$ on the complex plane we see that the principal argument of $-\sqrt{3} - i$ is $-5\pi/6$



Writing $z = r(\cos\theta + i\sin\theta)$ gives

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta) = 2(\cos(\frac{-5\pi}{6}) + i\sin(\frac{-5\pi}{6})).$$

Therefore $r = 2^{\frac{1}{5}}$ and $\theta = \frac{-\pi}{6} + \frac{2k\pi}{5}$, for any integer k. Since we know there are five fifth roots, we choose five values of k. We set $k = 0$, 1, 2, 3 and 4 in the equation

$$z = 2^{\frac{1}{5}}(\cos(\frac{-\pi}{6} + \frac{2k\pi}{5}) + i\sin(\frac{-\pi}{6} + \frac{2k\pi}{5}).$$

We obtain five different values of z:

$$z = 2^{\frac{1}{5}}\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right],$$
$$z = 2^{\frac{1}{5}}\left[\cos\left(\frac{7\pi}{30}\right) + i\sin\left(\frac{7\pi}{30}\right)\right],$$
$$z = 2^{\frac{1}{5}}\left[\cos\left(\frac{19\pi}{30}\right) + i\sin\left(\frac{19\pi}{30}\right)\right],$$
$$z = 2^{\frac{1}{5}}\left[\cos\left(\frac{31\pi}{30}\right) + i\sin\left(\frac{31\pi}{30}\right)\right],$$
$$z = 2^{\frac{1}{5}}\left[\cos\left(\frac{43\pi}{30}\right) + i\sin\left(\frac{43\pi}{30}\right)\right].$$

These are the five fifth roots of $-\sqrt{3} - i$. ◊

2.4 Roots of polynomial equations

We have already seen how to find roots of a complex number by understanding that the roots are solutions of a very simple type of polynomial equation. What can be said about solutions of more complicated polynomial equations such as $z^4 - 18z^2 + 192z - 175 = 0$? To discuss this more general type of equation we need to be clear about what we mean by a polynomial and how we can use different number sets in writing down and solving polynomial equations.

> A *polynomial* in z is an expression of the form
>
> $$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0$$
>
> where z is the variable and the numbers a_n, a_{n-1}, $\cdots$, a_0 are the coefficients.

If $a_n \neq 0$ then the polynomial is said to have *degree* n. The term $a_n z^n$ is known as the *leading term*. The *roots of a polynomial equation* are the numbers z which satisfy

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0 = 0.$$

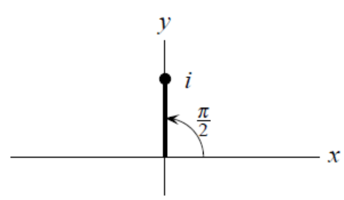
It is important to know in which number set the solutions of a polynomial equation lie. For example, the problem “solve $z^4-16=0$ over the real numbers” (or equivalently, “find the real roots of $z^4-16=0$”) has the answer $z=2$ or -2. If the problem is changed slightly to read “solve $z^4-16=0$ over the complex numbers” (or “find the complex roots of $z^4-16=0$”), then the correct answer is $z=2$, $2i$, -2 or $-2i$. Clearly this polynomial equation has more complex solutions than real solutions. It is a degree 4 polynomial equation and has four complex roots, although it has only two real roots.
In general

> A polynomial equation of degree n has at most n complex roots. All, some or none of these roots may be real.

✎ ***Example* 2.8.** Find all complex roots of the polynomial equation $z^5-iz^2=0$.
Observe that the left hand side can be factorised; this gives

$$z^2(z^3-i)=0.$$

Thus the roots of the original equation consist of all the roots of the equation $z^2=0$ together with all the roots of the equation $z^3-i=0$. The only root of $z^2=0$ is $z=0$. We now find the roots of$z^3-i=0$. We write i in polar form.



Clearly $|i|=1$ and $\arg i=\frac{pi}{2}+2k\pi$. As we are seeking cube roots, we expect three solutions, so we will use $k=0$, 1, 2 in the expression

$$1^{\frac{1}{3}}\left[\cos\left(\frac{\pi/2+2k\pi}{3}\right)+i\sin\left(\frac{\pi/2+2k\pi}{3}\right)\right].$$

When $k=0$ we obtain $\cos\pi/6+i\sin\pi/6$, when $k=1$ we obtain $\cos 5\pi/6+i\sin 5\pi/6$ and when $k=2$ we obtain $\cos 3\pi/2+i\sin 3\pi/2$. Therefore the complex roots of $z^3-i=0$ are $\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $\frac{\sqrt{-3}}{2}+\frac{1}{2}i$ and $-i$.
There are just four distinct roots of the original equation, namely

$$0,\ \frac{\sqrt{3}}{2}+\frac{1}{2}i,\ \frac{-\sqrt{3}}{2}+\frac{1}{2}i,\ -i.$$

$\diamondsuit$

☞ **[Aside]:**

Polynomial equations of degree n have at most n distinct complex roots, as some roots might be repeated. For example, the polynomial $z^2 - 2z + 1 = (z-1)^2$ has a *double root* at $z = 1$, also called a root of *multiplicity* 2. If n is a positive integer then the equation $z^n = 0$ has a root of multiplicity n at $z = 0$. It is true that every polynomial equation of degree n has exactly n complex roots, *counted with multiplicity*. ✌

We have already seen in Chapter 1 that when a quadratic equation with real coefficients has non-real complex roots, then these roots come in complex conjugate pairs. So for example, the equation $z^2 + 4z + 5 = 0$ has roots $z = -2 + i$ and $z = -2 - i$.
In fact:

> If the coefficients in a polynomial equation are *all real* then all of the no–real complex *roots occur* in complex conjugate pairs.

For example, the polynomial equation $z^4 - 16 = 0$ that was discussed earlier in this chapter has two real roots (2 and -2) and two imaginary roots ($2i$ and $-2i$), and these imaginary roots are complex conjugates of each other. The coefficients of this polynomial, 1 and -16 are both real and so we expect complex roots will occur in complex conjugate pairs. By contrast, the polynomial $z^3 - i = 0$, solved in Example 2.8 above, does not have all real coefficients; the coefficients are 1, which is real, and $-i$, which is not real. The roots of $z^3 - i = 0$ are $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\frac{-\sqrt{3}}{2} + \frac{1}{2}i$ and $-i$. Although they are all non–real complex numbers, they do not occur in complex conjugate pairs.
If one complex root of a polynomial equation with real coefficients is known then its complex conjugate can immediately be written down to give another root.

✎ ***Example* 2.9.** Find all roots of $z^4 - 18z^2 + 192z - 175 = 0$, given that $3 - 4i$ is a root.
If $3-4i$ is a root, then its complex conjugate $3+4i$ is also a root since the coefficients of the polynomial are real. We write down the quadratic expression with these two roots. We can then divide this quadratic into the original polynomial to get another quadratic which can be easily solved. Since $(z-(3-4i))(z-(3+4i)) = z^2 - 6z + 25$, we want to find $(z^4 - 18z^2 + 192z - 175)/(z^2 - 6z + 25)$.

Using polynomial long division:

$$
\begin{array}{r|rrrrr}
 & & & z^2 & +6z & -7 \\
\hline
z^2-6z+25) & z^4 & & -18z^2 & +192z & -175 \\
 & z^4 & -6z^3 & +25z^2 & & \\
\hline
 & & 6z^3 & -43z^2 & +192z & \\
 & & 6z^3 & -36z^2 & +150z & \\
\hline
 & & & -7z^2 & +42z & -175 \\
 & & & -7z^2 & +42z & -175 \\
\hline
 & & & & & 0
\end{array}
$$

Therefore we have

$$
\begin{aligned}
z^4-18z^2+192z-175 &= (z-(3-4i))\,(z-(3+4i))\left(z^2+6z-7\right) \\
&= (z-(3-4i))\,(z-(3+4i))\,(z-1)\,(z+7)\,.
\end{aligned}
$$

So the four roots of the original degree 4 polynomial are $3-4i$, $3+4i$, 1 and -7.

◊

☞ **[Aside]:**

It is not difficult to prove rigorously that if a polynomial equation with real coefficients has complex roots then these roots occur in complex conjugate pairs.

First we need to show that $\overline{z+w}=\overline{z}+\overline{w}$ and that $\overline{zw}=\overline{z}\overline{w}$.Try doing this by writing $z=a+ib$ and $w=c+id$ and calculating $\overline{z}+\overline{w}$ and $\overline{z}\overline{w}$.

Then, let us consider a polynomial

$$p(z)=a_nz^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\cdots+a_1z+a_0$$

where a_n, a_{n-1}, a_{n-2}, $\cdots$, a_0 are all real.

Suppose there is some complex number v which is a root of the polynomial, so that $p(v)=0$. If we take complex conjugates of both sides of the equation we have $\overline{p(\nu)} = \overline{0} = 0$ and hence

$$\begin{aligned}
0 &= \overline{p(\nu)} \\
&= \overline{a_n\nu^n + a_{n-1}\nu^{n-1} + a_{n-2}\nu^{n-2} + \cdots + a_1\nu + a_0} \\
&= \overline{a_n\nu^n} + \overline{a_{n-1}\nu^{n-1}} + \overline{a_{n-2}\nu^{n-2}} + \cdots + \overline{a_1\nu} + \overline{a_0} \\
&= \overline{a_n}\,\overline{\nu^n} + \overline{a_{n-1}}\,\overline{\nu^{n-1}} + \overline{a_{n-2}}\,\overline{\nu^{n-2}} + \cdots + \overline{a_1\nu} + a_0 \\
&= a_n\overline{\nu^n} + a_{n-1}\overline{\nu^{n-1}} + a_{n-2}\overline{\nu^{n-2}} + \cdots + a_1\overline{\nu} + a_0 \\
&= a_n\left(\overline{\nu}\right)^n + a_{n-1}\left(\overline{\nu}\right)^{n-1} + a_{n-2}\left(\overline{\nu}\right)^{n-2} + \cdots + a_1\overline{\nu} + a_0 \\
&= p(\overline{\nu})
\end{aligned}$$

Therefore since $p(\overline{\nu}) = 0$, $\overline{\nu}$ is a root. That is, both ν and its conjugate $\overline{\nu}$ are roots.
As you read through the above proof try to work out why each line follows from the previous line.

✌

✍ ***Exercises* 2.1.**

1. For each of the following numbers, give the numerical value of the real part x, the imaginary part y, the modulus r and the principal value of the argument θ . Plot the number as a point in the complex plane.

   a) $1 - i\sqrt{3}$ d) $2(\cos(\pi/6) + i\sin(\pi/6)$
   b) $1/(1-i)$ e) $\left(\frac{1+i}{1-i}\right)^2$
   c) $\left(i + \sqrt{3}\right)^2$ f) $\frac{3+i}{2+i}$

2. Write each of the following complex numbers in polar form.

$$-4i, \quad -2+2i \;\; 1-i$$

   a) $(-2+2i)(1-i)$ c) $(1-i)^6$
   b) $-4i/(-2+2i)$ d) $(-2+2i)^15$

3. Use de Moivre's theorem to simplify

   a) $(\cos(2\pi/3) + i\sin(2\pi/3))^9$ c) $(\cos(2\pi/3) - i\sin(2\pi/3))^6$
   b) $(\cos(\pi/3) + i\sin(\pi/3))^4$ d) $(\sin(2\pi/3) + i\cos(2\pi/3))^9$.

4. Recall that if $p(z)$ is a polynomial with real coefficients and if $w \in \mathbb{C}$ is a root of $p(z)$ then so is $\overline{w}$. Find the roots of the quadratic equation $q(z) = z^2 - 3(1+i)z - 2 + 6i = 0$. Verify that if

w is a root of $q(z)$ then $\overline{w}$ is not a root and explain why this does not contradict the statement at the start of this question.

5. Find all the roots of $f(z) = z^4 - 3z^3 + 7z^2 + 21z - 26$, given that $2 - 3i$ is a root.

6. Find all the roots of $z^4 - 5z^3 + 4z^2 + 2z - 8$, given that $1 - i$ is a root.

Chapter 3 Polar exponential form

A detailed list of mathematical objectives (knowledge, understanding and skills) for this chapter is provided in the weekly tutorial exercises. Equally important are the *generic skills you will develop. The study of mathematics enhances your ability to think logically and analytically, move from the particular to the general, work quantitatively and improve problem-solving skills*. By reading and working carefully through the material in this chapter you will develop the following additional generic skills:

- Recognise the same information when presented in different forms.
- Develop the ability to choose the form most suitable for the task at hand.
- Convert objects from one form to another.
- Use geometric/visual techniques to help understand new concepts.
- Generalise ideas from simple and familiar settings to more abstract settings.

In this chapter a more concise polar form of a complex number, called polar exponential form, is introduced; we see how to use this new form to derive some interesting trigonometric relationships. Polar exponential form also leads to the definition of a new function, namely the complex exponential function. Since functions are fundamental to the study of all branches of mathematics, this provides an opportunity to review the definition of a function and discuss some of the important properties of functions.

3.1 Polar exponential form

Consider two complex numbers of modulus 1 given in polar form. In the previous chapter we saw that when multiplying, we add arguments; when dividing, we subtract arguments.
This process is reminiscent of the way we manipulate powers of the same base in the real number system. Recall that when multiplying powers of a positive number a, we add the powers:

$$a^x \times a^y = a^{x+y}$$

and when dividing we subtract the powers:

$$a^x/a^y = a^{x-y},$$

where a, x, y are real numbers and $a > 0$.
This leads us to make use of the same type of notation for complex numbers in polar form.

***Theorem* 3.1.** (*Euler formula*)

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \forall\,\theta \in \mathbb{R}. \tag{3.1.1}$$

Proof: Recall the Taylor expansions for the functions e^x, $\cos x$ and $\sin x$ about $x = 0$, with $x \in \mathbb{R}$:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots, && (3.1.2)\\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, && (3.1.3)\\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, && (3.1.4)
\end{aligned}$$

in agreement to our knowledge that $\cos x$ is an even function, and $\sin x$ an odd one. Then, substituting $x = i\theta$ into e^x, and noticing $i^2 = -1, i^3 = -i$ and $i^4 = 1$, we have

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \tfrac{(i\theta)^2}{2!} + \tfrac{(i\theta)^3}{3!} + \tfrac{(i\theta)^4}{4!} + \tfrac{(i\theta)^5}{5!} + \tfrac{(i\theta)^6}{6!} + \cdots\\
&= 1 + i\theta - \tfrac{\theta^2}{2!} - i\tfrac{\theta^3}{3!} + \tfrac{\theta^4}{4!} + i\tfrac{\theta^5}{5!} - \tfrac{\theta^6}{6!} - \cdots\\
&= 1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - \tfrac{\theta^6}{6!} + \cdots\\
&\quad + i\left(\theta - \tfrac{\theta^3}{3!} + \tfrac{\theta^5}{5!} - \cdots\right),
\end{aligned}$$

hence in comparison with eqs.(3.1.3) and (3.1.4) we recognize

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Obviously, $\cos\theta$ accounts for the even part of $e^{i\theta}$, and $\sin\theta$ the odd part.

$\square$

[Remarks]:

- $e^{i\theta}$ has argument θ, and modulus 1, due to $|e^{i\theta}|= \sqrt{\cos^2\theta + \sin^2\theta} = 1$. For any value of θ, the complex number $e^{i\theta}$ lies on the unit circle $|z|= 1$.

- From (3.1.1) we have

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta. \tag{3.1.5}$$

  Hence

$$\cos\theta = \frac{e^{i\theta} + e^{i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i}. \tag{3.1.6}$$

- If θ and ϕ are any two real numbers, calculations done in the previous chapter show us that

$$e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi) = e^{i(\theta+\phi)}, \tag{3.1.7}$$

  and

$$\frac{e^{i\theta}}{e^{i\phi}} = \frac{(\cos\theta + i\sin\theta)}{(\cos\phi + i\sin\phi)} == \cos(\theta-\phi) + i\sin(\theta-\phi) = e^{i(\theta-\phi)}. \tag{3.1.8}$$

This new notation allows us to write a complex number of modulus r and argument θ in a more concise polar form called *polar exponential form*,

$$z = re^{i\theta}$$

We now have three ways of writing a non–zero complex number z: [*1]

- Cartesian form: $z = a + ib$;

- Polar form: $z = r(\cos\theta + i\sin\theta)$;

- Polar exponential form: $z = re^{i\theta}$.

☞ **[Aside]:**

When substituting $\theta = \pi$ into $e^{i\theta} = \cos\theta + i\sin\theta$, we obtain $e^{i\pi} = \cos\pi + i\sin\pi$, or

$$e^{i\pi} + 1 = 0.$$

It is remarkable because one equation of great simplicity contains five important constants: e, θ, i, 1 and 0. ✌

[*1] The zero complex number $0 = 0 + i0$ is not assigned a polar form as there is no sensible way to assign it an argument.

3.2 Arithmetic in polar exponential form

Multiplication and division

Polar exponential form, by design, has the same useful property of ordinary polar form; when multiplying or dividing two numbers in polar exponential form the answer is automatically expressed in polar exponential form as well, allowing the new modulus and argument to be read off immediately. Let $z = re^{i\theta}$ and $w = se^{i\phi}$ be any two non–zero complex number. Then

$$zw = re^{i\theta} \times se^{i\phi} = rse^{i(\theta+\phi)}$$

and

$$\frac{z}{w} = \frac{re^{i\theta}}{se^{i\phi}} = \frac{r}{s}e^{i(\theta-\phi)}.$$

✎ ***Example* 3.1.** Find z^2w^3 and z^2/w^3 when $z = 2e^{i\frac{\pi}{4}}$ and $w = 3e^{i\frac{3\pi}{2}}$.
We first calculate z^2 and w^3, to obtain

$$z^2 = 4e^{i\frac{\pi}{2}}$$

and

$$w^3 = 27e^{i\frac{9\pi}{2}}.$$

Therefor

$$z^2w^3 = 4e^{i\frac{\pi}{2}} \times 27e^{i\frac{9\pi}{2}} = 108e^{5\pi i} = -108.$$

Similarly,

$$\frac{z^2}{w^3} = \frac{4e^{i\frac{\pi}{2}}}{27e^{i\frac{9\pi}{2}}} = \frac{4}{27}e^{-4\pi i} = \frac{4}{27}.$$

◊

Raising to an integer power

For every positive integer n, we have

$$z^n = \left(re^{i\theta}\right)^n = \left(re^{i\theta}\right) \times \left(re^{i\theta}\right) \times \cdots \times \left(re^{i\theta}\right) = r^n e^{in\theta}.$$

In fact, this hold for all integers n, whether positive, negative or zero.

✎ ***Example* 3.2.** Using polar exponential form, find z^8 when $z = 1 + \sqrt{3}i$.
We find that $|z| = 2$ and arg $z = \frac{\pi}{3}$, so $z = 2e^{\frac{\pi}{3}i}$. Hence

$$z^8 = 2^8 e^{\frac{8\pi}{3}i} = 256e^{\frac{8\pi}{3}i} = 256e^{\frac{2\pi}{3}i} = -128 + 128\sqrt{3}i.$$

◊

✎ ***Example* 3.3.** Simplify $e^{\frac{15\pi}{7}i}$.

We have

$$\begin{aligned} e^{\frac{15\pi}{7}i} &= e^{2\pi i+\frac{\pi}{7}i} = e^{2\pi i} \times e^{\frac{\pi}{7}i} == (\cos 2\pi + i\sin 2\pi)\, e^{\frac{\pi}{7}i} \\ &= (1+i0)\, e^{\frac{\pi}{7}i} = e^{\frac{\pi}{7}i}. \end{aligned}$$

$\Diamond$

The last example demonstrates once again that arguments are only determined up to integer multiples of 2π; that is, $e^{i\theta}$ is the same as $e^{i\phi}$ when θ and ϕdiffer by an integer multiple of 2π. It is useful to remember that $e^{2\pi i} = 1$ and that for every integer n, $e^{2n\pi i} = 1$.

If $re^{i\theta} = se^{i\phi}$, then $r = s$ and $\theta = \phi + 2k\pi$, for some integer k.

3.3 Cosine and sine in terms of exponentials

We have, for all real θ ,

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3.3.9}$$

and so

$$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta. \tag{3.3.10}$$

Adding the above two equations we obtain

$$e^{i\theta} + e^{-i\theta} = \cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta) = 2\cos\theta,$$

which rearranges to give

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right).$$

Subtracting $e^{-i\theta}$ from $e^{i\theta}$ gives

$$e^{i\theta} - e^{-i\theta} = e^{i(-\theta)} = \cos(\theta) + i\sin(\theta) - \cos(\theta) + i\sin(\theta) = 2i\sin\theta$$

which rearranges to give

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

They are useful formulas:

> For all real θ, $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

These expressions for $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ can be used to derive some special trigonometric identities which are helpful in solving certain types of integration problems involving powers of $\cos$ and $\sin$, as we shall see soon. Since these involve the binomial theorem, we first revise that theorem very briefly.

Binomial theorem

The expression $x + y$ is called a binomial expression, and the binomial theorem is a generalisation of the familiar formula $(x + y)^2 = x^2 + 2xy + y^2$. It states that for all x, y and for all integers $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n$$

Using ${}^nC_r$ notation, where ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + \cdots + {}^nC_rx^{n-r}y^r + \cdots + {}^nC_ny^n$$

The numbers $\binom{n}{0}$, $\binom{n}{1}$, $\cdots$,$\binom{n}{n}$, are called the *binomial coefficients*.

Thus, for example, with $n = 4$ we have $(x + 4)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, and with $n = 5$ we have $(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

The binomial coefficients can also be found in *Pascals Triangle*, part of which is shown below:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Note that each row gives the binomial coefficients for a particular value of n in the expansion of $(x + y)^n$. For example, in the expansion of $(x + y)^3$ the coefficients are 1, 3, 3, 1. Another interesting feature of Pascals triangle is that each entry is the sum of the two entries in the row above it. In the last row shown above, $21 = 15 + 6$, $35 = 20 + 15$, and so on. So you can add on as many additional rows as you require, remembering that each row begins and ends with a 1, to enlarge the triangle to any n value you please.

Applications to trigonometry

Lets now calculate $\cos^3\theta$ using the binomial theorem with $n = 3$ (binomial coefficients 1, 3, 3, 1) and the fact that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

$$\begin{aligned}
\cos^3\theta &= \frac{1}{8}\left(e^{i\theta} + e^{-i\theta}\right)^3 \\
&= \frac{1}{8}\left(e^{3i\theta} + 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} + e^{-3i\theta}\right) \\
&= \frac{1}{8}\left(e^{3i\theta} + 3e^{3i\theta} + 3e^{-i\theta} + e^{-3i\theta}\right) \\
&= \frac{1}{4}\left(\frac{1}{2}\left(e^{3i\theta} + e^{-3i\theta}\right) + \frac{3}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right) \\
&= \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)
\end{aligned}$$

This formula for $\cos^3\theta$ may also be calculated simply using standard trigonometric identities, but the working is not as elegant as this approach. The same technique can be used to find expressions for any positive power of $\cos\theta$ and $\sin\theta$, which can then be used in integration problems.

✎ ***Example* 3.4.** Find $\int \cos^3\theta\,\mathrm{d}\theta$.

$$\begin{aligned}
\int \cos^3\theta\,\mathrm{d}\theta &= \int \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)\,\mathrm{d}\theta \\
&= \frac{1}{4}\int \cos 3\theta\,\mathrm{d}\theta + \frac{3}{4}\int \cos\theta\,\mathrm{d}\theta \\
&= \frac{1}{12}\sin 3\theta + \frac{3}{4}\sin\theta + C,
\end{aligned}$$

where C is an arbitrary constant. $\Diamond$

✎ ***Example* 3.5.** Find a formula for $\sin^4\theta$ in terms of $\cos 4\theta$ and $\cos 2\theta$.

Using the binomial theorem with $n = 4$ and the expression for $\sin\theta$ in terms of exponentials, we obtain

$$\begin{aligned}
\sin^4\theta &= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^4 \\
&= \frac{1}{16i^4}\left(e^{4i\theta} - 4e^{3i\theta}e^{-i\theta} + 6e^{2i\theta}e^{-2i\theta} - 4e^{i\theta}e^{-3i\theta} + e^{-4i\theta}\right) \\
&= \frac{1}{16}\left(e^{4i\theta} - 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6\right) \\
&= \frac{1}{8}\left(\frac{1}{2}\left(e^{4i\theta} + e^{-4i\theta}\right) - 4 \times \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right) + 3\right) \\
&= \frac{1}{8}\left(\cos 4\theta - 4\cos 2\theta + 3\right)
\end{aligned}$$

◊

So far we have shown how to obtain formulas for powers of $\sin n\theta$ and $\cos n\theta$ in terms of cosines of multiples of θ , but we can also reverse the process to find formulas for expressions like $\cos\theta$ and $\sin\theta$. In these cases we revert to the non-exponential polar form and apply both de Moivres theorem and the binomial theorem to $(\cos\theta + i\sin\theta)^n$.

✎ ***Example* 3.6.** Find formulas for $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4\\ &= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\left(i^2\sin^2\theta\right) + 4\cos\theta\left(i^3\sin^3\theta\right) + i^4\sin^4\theta\\ &= \cos^4\theta - 6\cos^2\sin^2\theta + \sin^4\theta + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^{3\theta}\right)\end{aligned}$$

Equating the real parts on both sides of the equation gives

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

and equating the imaginary parts gives

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$$

◊

3.4 Complex exponential function

We have defined $e^{i\theta}$ as $\cos\theta + i\sin\theta$ for any $\theta \in \mathbb{R}$. This has given us a way of calculating complex exponentials where the exponent is a purely imaginary number. So, for example,

$$e^{3i} = \cos 3 + i\sin 3 \approx -0.989 + 0.141i.$$

(We are using radian measure, not degrees.)

The next step is to extend this so we can define exponentials of any complex number. Let $z = x + iy$ be a complex number expressed in Cartesian form. What would be a sensibleway of defining e^z? Weve already seen that expressions like $e^{i\theta}$ are manipulated using the standard rules that apply to real exponentials: add exponents when multiplying, subtract exponents when dividing and, when taking integer powers, multiply exponents by that integer. It would be sensible to define e^z so that these familiar and easy–to–use rules also apply in this general setting.

***Definition* 3.1.** If $z = x + iy$ with x and y real, we define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Study this definition carefully, especially the last expression:

$$e^z = e^x(\cos y + i \sin y).$$

Notice that since x and y are real and e^x is real and positive, this displays e^z as a complex number in *polar form*. We can therefore read off the modulus and argument of e^z.

When $z = x + iy$ with x and y real, $|e^z|= e^x$ and $\arg e^z = y$.

The usual rules formultiplying, dividing and taking integer powers still apply. If $z = x + iy$ and $w = \mu + i\nu$ we know that $z + w = (x + \mu) + i(y + \nu)$, $z - w = (x - \mu) + i(y - \nu)$ and $nz = nx + iny$. Hence

$$e^z \times e^w = \left(e^x e^{iy}\right)\left(e^{\upsilon} e^{i\nu}\right) = e^{x+\upsilon} e^{i(y+\nu)} = e^{(z+w)},$$

$$\frac{e^z}{e^w} = \frac{e^x e^{iy}}{e^{\upsilon} e^{i\nu}} = e^{x-\upsilon} e^{i(y-\nu)} = e^{(z-w)},$$

and for any integer n,

$$\left(e^z\right)^n = \left(e^x e^{iy}\right)^n = \left(e^x\right)^n \left(e^{iy}\right)^n = e^{nx} e^{iny} = e^{nz}.$$

We can now calculate the value of e^z for any complex number z.

✎ ***Example* 3.7.** Express the complex exponentials e^0, e^{2+4i}, $e^{-1+i\pi/4}$, $e^{-1+i17\pi/4}$ and e^x where x is real, as complex numbers in Cartesian form.

Solution: Using the definition of e^z, we obtain

$$\begin{aligned}
e^0 &= e^{0+0i} = e^0\left(\cos 0 + i \sin 0\right) = 1\left(1 + i0\right) = 1, \\
e^{2+4i} &= e^2\left(\cos 4 + i \sin 4\right) = e^2 \cos 4 + ie^2 \sin 4 \approx -4.83 - 5.59i, \\
e^{-1+i\pi/4} &= e^{-1}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right) = e^{-1}\cos\frac{\pi}{4} + ie^{-1}\sin\frac{\pi}{4} \approx 0.26 + 0.26i, \\
e^{-1+i17\pi/4} &= e^{-1}\left(\cos\frac{17\pi}{4} + i \sin\frac{17\pi}{4}\right) = e^{-1}\cos\frac{17\pi}{4} + ie^{-1}\sin\frac{17\pi}{4} \approx 0.26 + 0.26i, \\
e^x &= e^{x+i0} = e^x\left(\cos 0 + i \sin 0\right) = e^x\left(1 + i0\right) = e^x.
\end{aligned}$$

That the third and fourth results are equal should be no surprise, since $\pi/4$ and $17\pi/4$ differ by an integer multiple of 2π and from previous work, $e^{2\pi i} = 1$. The last result shows that when z equals the real number x, the complex expression e^z agrees with the usual real exponential e^x. ♢

What we have done in this section is to set out a way of calculating the value of e^z for any complex number z. The value of e^z is also a complex number. You can probably already see that we have in fact defined a *function*, with the set of all complex numbers $\mathbb{C}$ as the set of inputs and the outputs also being members of the set $\mathbb{C}$. How does this tie in with your notion of function from high school calculus?

Chapter 4 Fourier series and transform (I) — Periodic functions

4.1 Summary of formulae in Chapters 4 and 5

Let $f(x)$ be a periodic function over $[-\pi, \pi]$. The Fourier series expansion of $f(x)$ is conventionally given by

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + \sum_{n=1}^{\infty} B_n \sin nx \tag{4.1.1}$$

where

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \\ A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \\ B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx. \end{aligned}$$

Here $\{\cos mx, \sin nx\}$, $m, n \in \mathbb{Z}$, forms an orthonormal basis, satisfying

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = 0 \quad (m \neq n), \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(nx)dx = 1, \\ &\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0, \quad \forall m, n. \end{aligned} \tag{4.1.2}$$

The vector space spanned by this basis is called a Hilbert space.
If $f(x)$ is a periodic function over $[-L, L]$, instead of $[-\pi, \pi]$, the corresponding Fourier series is given by

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \tag{4.1.3}$$

where

$$
\begin{aligned}
A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, \\
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \\
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.
\end{aligned}
$$

If the function $f(x)$ is defined on the interval $[0, 2L]$, the above equations simply become

$$
\begin{aligned}
A_0 &= \frac{1}{2L}\int_{0}^{2L} f(x)dx, \\
A_n &= \frac{1}{L}\int_{0}^{2L} f(x)\cos\frac{n\pi x}{L}dx, \\
B_n &= \frac{1}{L}\int_{0}^{2L} f(x)\sin\frac{n\pi x}{L}dx.
\end{aligned}
$$

In the light of the Euler formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

the expansion onto the basis $\{\cos mx, \sin nx\}$, $m, n \in \mathbb{Z}$, can also be turned into an expansion onto the basis $\{e^{inx}\}$, $n \in \mathbb{Z}$. The exponential form of Fourier series with x is

$$f(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx}, \qquad m \in \mathbb{Z}, \tag{4.1.4}$$

where

$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx. \tag{4.1.5}$$

The Fourier transform, also called the Fourier integral, is mathematically stated as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega, \tag{4.1.6}$$

$$G(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \tag{4.1.7}$$

where $G(\omega)$ is called the kernel function of the Fourier transform.

[Remark]: ω is a dual variable of t, and ωt guarantees to be dimension-less.

✎ ***Example* 4.1.** Find the Fourier series of the function $f(x)$ defined by $f(x) = x^2$ on the interval [-2, 2].

Solution: Since this function is an even function on the given interval (where $L = 2$), it follows that its Fourier series is a pure cosine series. Thus, its Fourier coefficients B_n vanish. In this case these integrals become

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{2}\int_0^2 x^2 dx = \frac{4}{3} \\ A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \\ &= \int_0^2 x^2\cos\frac{n\pi x}{2}dx = 16\frac{(-1)^n}{n^2\pi^2}, \qquad \forall n \in \mathbb{Z},\ n \geq 1. \end{aligned}$$

From these coefficients we get the Fourier series of x^2

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = \frac{4}{3} + \sum_{n=1}^{\infty} 16\frac{(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

for $-2 \leq x \leq 2$. ◊

✎ ***Example* 4.2.** Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 0, & \text{when } -\pi < x \leq 0, \\ x, & \text{when } 0 < x \leq \pi. \end{cases}$$

Try to find the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$.

Solution:

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{2\pi}\int_0^{\pi} f(x)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} 0dx + \frac{1}{2\pi}\int_0^{\pi} xdx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{2\pi}\left(\frac{\pi^2}{2} - 0\right) = \frac{\pi}{4}, \end{aligned}$$

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nxdx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nxdx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 0\cos nxdx + \frac{1}{\pi}\int_0^{\pi} x\cos nxdx = \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx\right\} \\ &= \frac{1}{\pi}\left\{\left(\pi\frac{\sin n\pi}{n} - 0\right) - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left\{\frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right\} = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]. \end{aligned}$$

Obviously, this result depends on the parity of n, i.e., depends on n is an odd or even number.

Similarly, we can compute $B_n = -\frac{1}{n}(-1)^n,\ \forall n \geq 1$. We now have

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx) \\ &= \frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ [(-1)^n - 1] \cos nx - (-1)^n n \sin nx \right\}. \end{aligned}$$

$\Diamond$

✎ ***Example* 4.3.** Find the Fourier transform of the following function, in terms of the exponential form:

$$f(x) = \begin{cases} x, & \text{when } |x| \leq 1, \\ 0, & \text{when } |x| > 1. \end{cases}$$

Solution:

$$G(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-1}^{1} x e^{-ikx} dx = \frac{2i}{k} \cos k - \frac{2i}{k^2} \sin k.$$

Then the Fourier integral of $f(x)$ is written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(k) e^{ikx} dk = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{k \cos k - \sin k}{k^2} e^{ikx} dk.$$

$\Diamond$

4.2 Time-domain descriptions

Considerable functions are able to be analysed by signal theory in terms of time and frequency, among which a typical category are periodic signals. A periodic signal is a function which repeats itself every T seconds, where T is called the period of the signal. Portions of such periodic signals are illustrated in Figure 4.1. Achieving a clear analysis and understanding of the periodic signals is absolutely beneficial for our further study of aperiodic and random signals.

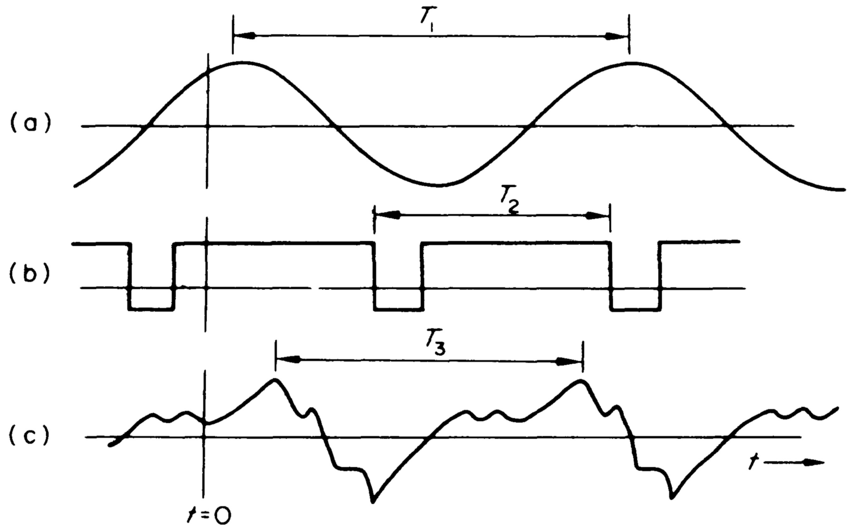


Figure 4.1: Portions of three periodic waveforms. (a), (b) and (c) are of the periods T_1, T_2 and T_3, respectively.

A complete time-domain description of such a signal involves specifying its value precisely at every instant of time. In some cases this may be done very simply using mathematical notation; for example, waveform (a) of figure 4.1 is completely specified by the function

$$f(t) \quad = A \sin\left(\omega t + \phi\right). \tag{4.2.8}$$

Waveform (b) is also quite simple to express mathematically, whereas (c) is obviously highly complex. If it is desired to approximate the signal by a mathematical expression, such techniques as a polynomial expansion, a Taylor series, or a Fourier series may be useful.

Revisit to Taylor series

The generic expression of a polynomial of order n reads

$$f(t) \quad = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n, \qquad a_0, a_1, \cdots, a_n \text{ — coefficients.} \tag{4.2.9}$$

It can be used to fit an actual curve at $(n+1)$ arbitrary points, as shown in figure 4.2. The accuracy of fitting generally improves as the number of polynomial terms increases. This polynomial approximation fits the actual waveform at a certain number of points. The alternative Taylor series approximation provides a good fit to a smooth continuous waveform in the vicinity of one selected point; i.e., the coefficients of the Taylor series are chosen to make the series and its derivatives agree with the actual waveform at the point. The number of terms in the series determines *to what order of derivative this agreement extends*, and hence *the accuracy with which the series and the actual waveform agree in the neighborhood of the point chosen*. The general form of the Taylor series for approximating a function $f(t)$ about a point $t = a$ is given by

$$f(t) \simeq f(a) + \frac{df(a)}{dt}(t-a) + \frac{1}{2!}\frac{d^2 f(a)}{dt^2}(t-a)^2 + \cdots + \frac{1}{n!}\frac{d^n f(a)}{dt^n}(t-a)^n. \tag{4.2.10}$$

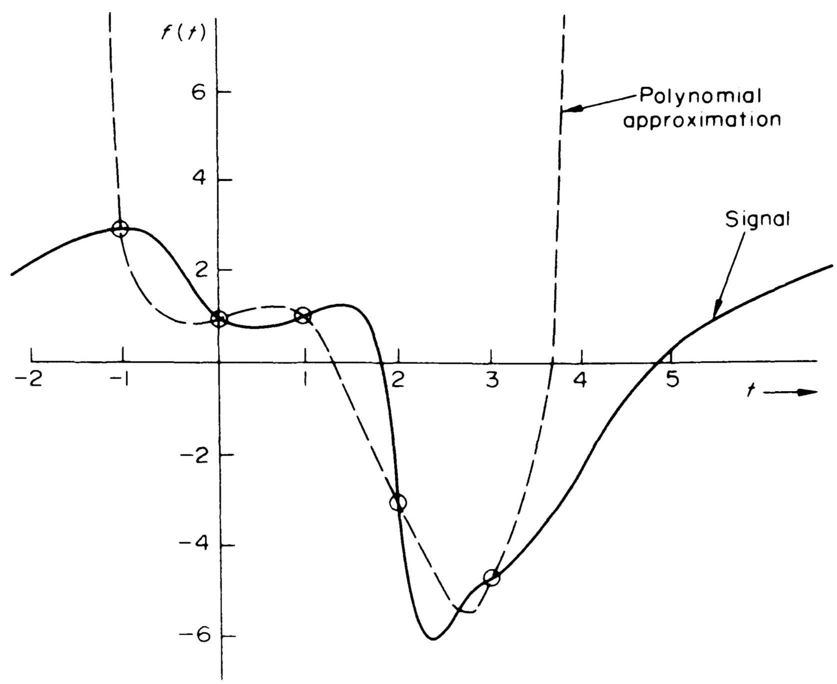


Figure 4.2: Time-domain approximation of a signal waveform in terms of a polynomial. The function $f(t) = (1 + t + 0.5t^2 - 2t^3 + 0.5t^4)$ is fit to a signal by five points.

A simple example is illustrated in figure 4.3, where the sinusoidal wave $(\sin t)$ is approximated about $t = \frac{\pi}{6}$ by the first three terms of a polynomial. With the period of the wave chosen as 1 second, the polynomial reads

$$0.5 + 5.44(t - 0.0833) - 19.7(t - 0.0833)^2.$$

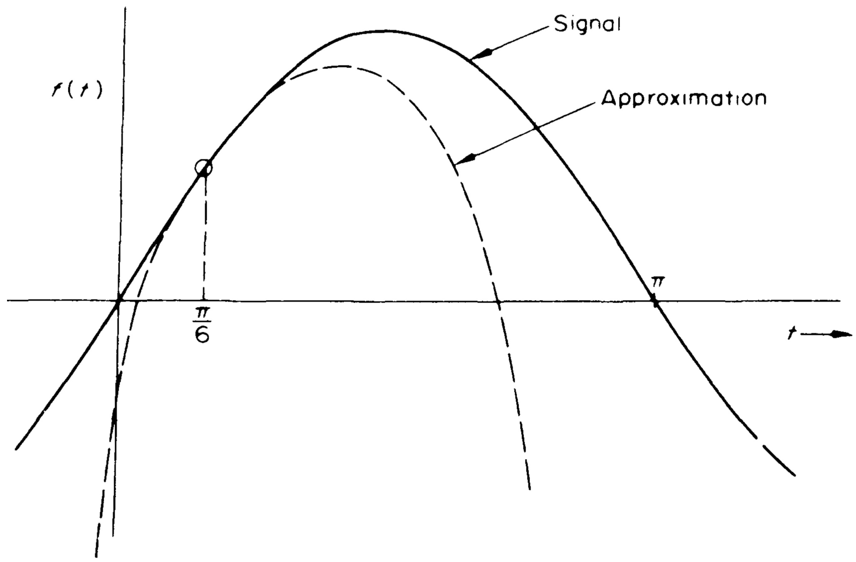


Figure 4.3: Approximation of a signal by a Taylor series. The first three terms of the series have been used to approximate the function ($\sin t$) about the point $t = \pi/6$.

Advantage and disadvantage

As would be expected, the fit to the actual waveform is acceptable in the neighborhood of the point chosen, but rapidly deteriorates to either side. The polynomial and Taylor series descriptions of a signal waveform are therefore recommendable when one is concerned to achieve accuracy over a limited region of the waveform. The accuracy usually decreases rapidly away from this region, although it might be improved by adding more expansion terms. The approximations provided by such methods are not particularly periodic in form, i.e., not in particular for describing repetitive signals.

4.3 Frequency-domain descriptions

In contrast with the Taylor series approximation, the Fourier series approximation is well suitable for describing periodic a signal in terms of sinusoidal functions which are periodic themselves.

A basic concept of frequency-domain analysis is that a waveform of any complexity may be approximated by the sum of a certain number of sinusoidal waveforms of suitable amplitude, periodicity and relative phase. A continuous sinusoidal function ($\sin \omega t$) is thought of as a *single frequency* wave of frequency ω in the unit of radians/second. The frequency-domain description of a signal presents a decomposition of a number of basic functions of such kind, which gives the method of Fourier analysis. In figure 4.4 illustrated is a periodic wave built up from a number of sinusoidal waves. The periodic waveform chosen is of 'sawtooth' form; its Fourier series is given by the summation of an infinite number of sinusoidal waves:

$$f(t) = \sin \omega t - \frac{1}{2} \sin 2\omega t + \frac{1}{3} \sin 3\omega t - \frac{1}{4} \sin 4\omega t + \cdots . \tag{4.3.11}$$

The sawtooth wave contains the frequencies:

- ω — known as the *fundamental component*,
- 2ω — the *second harmonic*,
- 3ω — the *third harmonic*; and so on,

with the amplitudes decreasing according to the increasing frequencies.

The approximation shown in figure 4.4 is obtained through summing up the first 4 terms of the series. If we wish to synthesize a sawtooth waveform perfectly, it is necessary to sum up an infinite number of terms in the series — in particular, sudden changes of the waveform are produced by very high frequency terms.

Frequency spectrum

An alternative way to graphically give the frequency-domain description for a sawtooth waveform is the so-called *frequency spectrum*, in the space of frequency. This is shown in figure 4.5, where the wave is consisted of a sinusoidal wave with frequency ω_1 and amplitude 1, a sinusoidal wave owith frequency $2\omega_1$ and amplitude $\frac{1}{2}$, a sinusoidal wave with frequency $3\omega_1$ and amplitude $\frac{1}{3}$, and so forth.

A important reason why sinusoidal functions are so crucial in signal analysis is that they occur widely in our physical world — *e.g.,* simple harmonic motion, vibrating strings and structures, wave motion, alternating electrical current, etc..

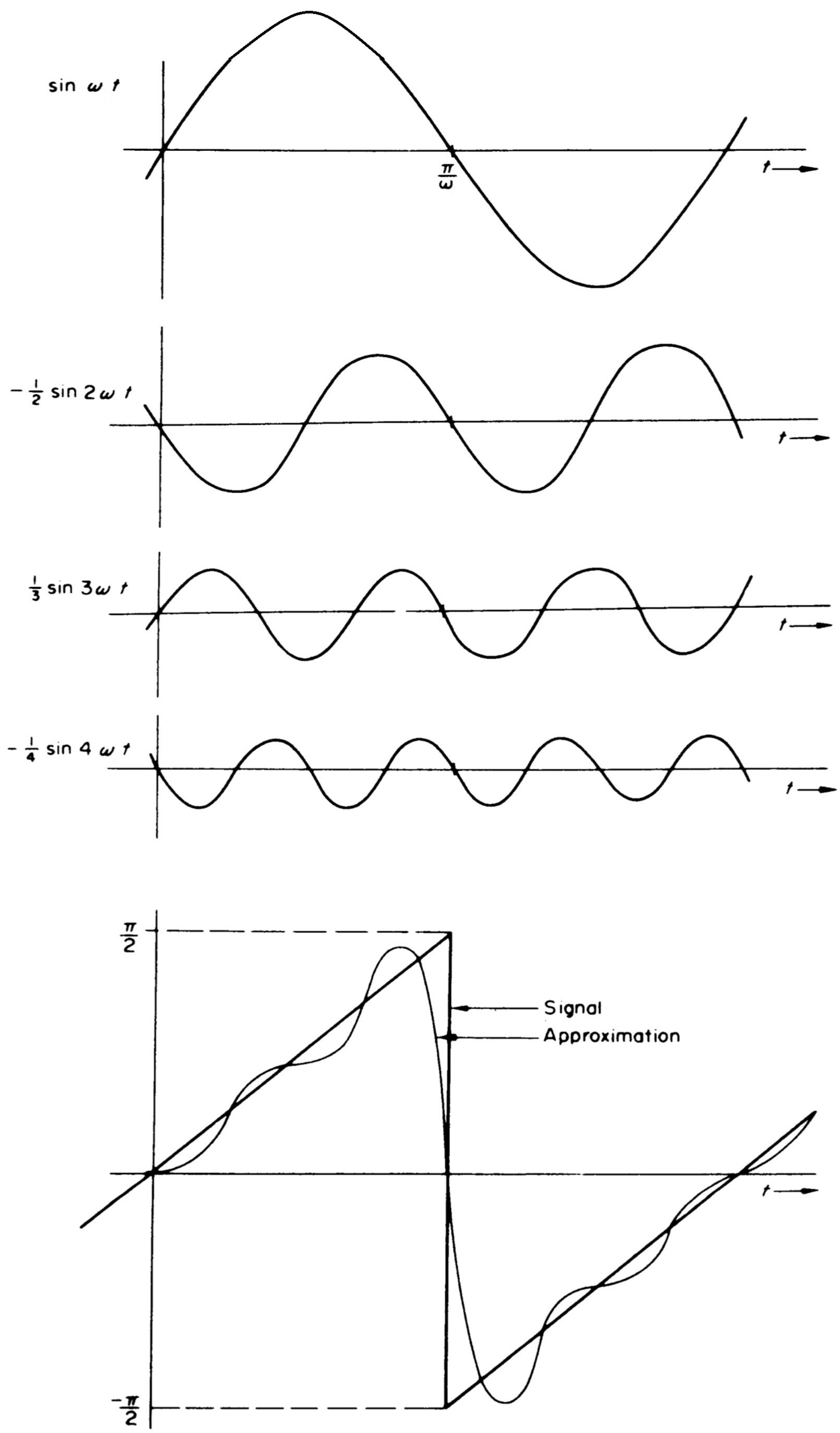


Figure 4.4: Synthesis of a periodic signal of 'sawtooth' form by the addition of a number of sinusoidal functions.

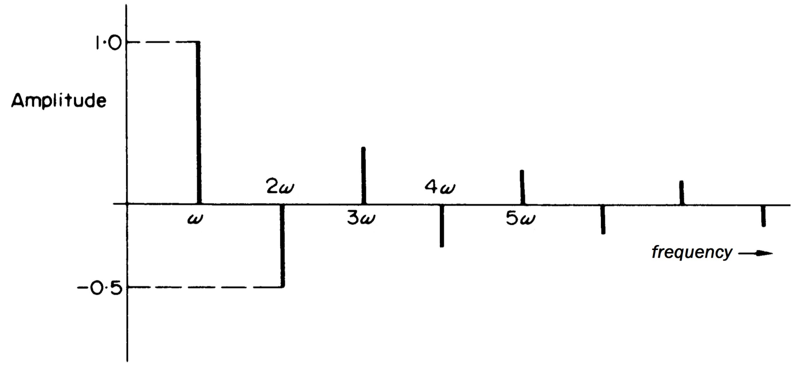


Figure 4.5: The frequency spectrum of the sawtooth wave illustrated in Figure 4.4.

☞ **[Aside]:**

Signal analysis in sinusoidal terms is of practical importance since the sinusoidal functions carry substantial physical meaning. A typical demonstration is the following experiment of optical Fourier transform, where a filtering in the frequency space is performed.

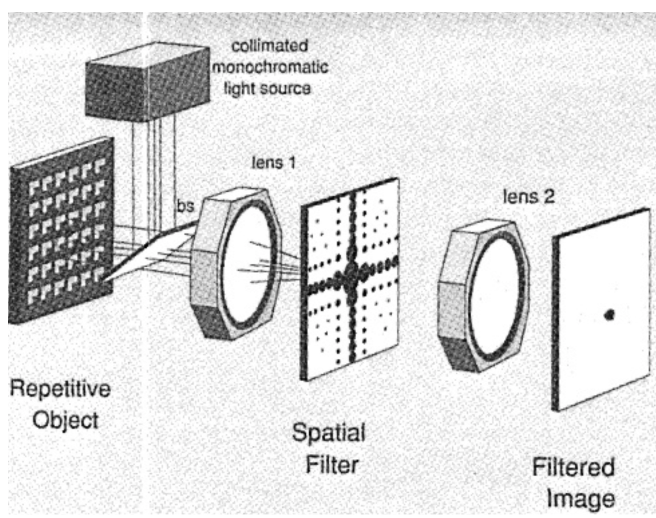


Figure 4.6: Optical Fourier transform: The region outside the Lens 1 and 2 is our real space; the region in between is the frequency space. The light sending from the object travels to Lens 1 and then is Fourier transformed according to frequency. On the screen there are filtering holes which block the light of some particular frequencies. Then the remaining light keeps traveling to Lens 2 and then forms the image, after undergoing an inverse Fourier transform.

Usually a signal is composed of low- and high-frequency components, as shown in Figure 4.7.

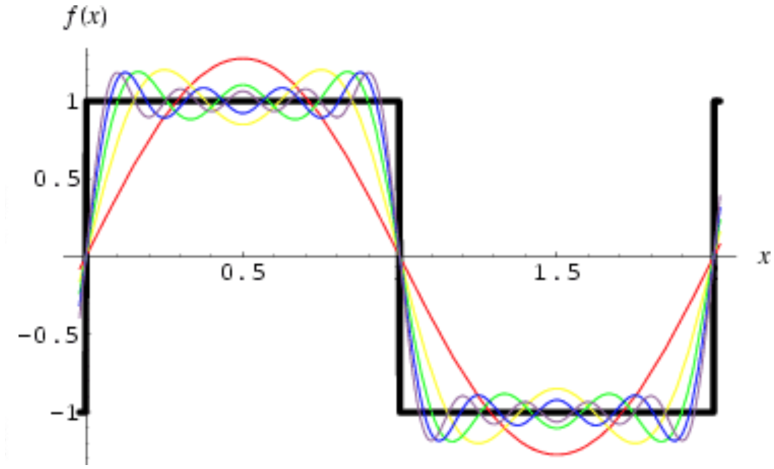


Figure 4.7: A square wave signal is composed of low-frequency and high-frequency components.

Then an original image may be turned into different images when filtering off the high-frequency components or the low-frequency components, as shown in Figure 4.8.

Original image | Low-pass filtering | High-pass filtering

Figure 4.8: The *left* image is an original image. It is turned into the *middle* image when filtering off the high-frequency components, i.e., letting the low-ones pass through; it is turned into the *right* image when filtering off the low-frequency components, i.e., letting the high-ones pass through.

Generally speaking, low-frequency components are responsible for the coarse sketch of the image, while high-frequency ones are for the sharp details of the image. In image processing this is called an adjustment of the *contrast*.

✌

4.4 Orthogonal functions

4.4.1 Vectors and signals

A discussion of orthogonal functions for describing signals may be introduced by considering the analogy between signals and vectors. A vector is specified both by its magnitude and direction, familiar examples being force and velocity. Consider a vector $\mathbf{v}$ in a two-dimensional space as shown in figure 4.9. Let and $\mathbf{v}_1$ and and $\mathbf{v}_2$ be two orthogonal vectors in the space. Geometrically, the component of $\mathbf{v}$ in the direction of $\mathbf{v}_1$ is obtained via the construction:

$$\mathbf{v} = C_1\mathbf{v}_1 + \mathbf{v}_e, \qquad \text{i.e.,} \qquad \mathbf{v}_e = \mathbf{v} - C_1\mathbf{v}_1. \tag{4.4.12}$$

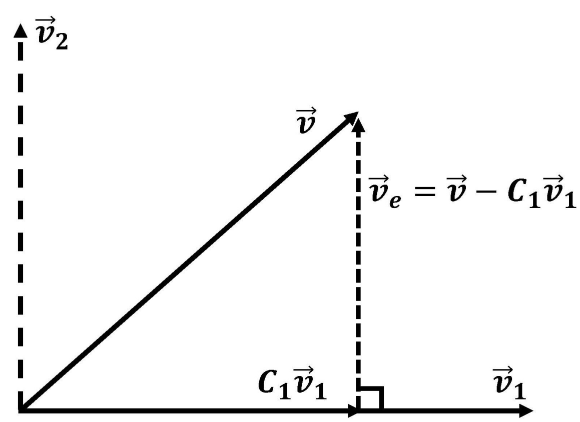


Figure 4.9: Let $\mathbf{v}$ be a vector in a two -dimensional space, which is endowed with two orthogonal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. If we use the projection of $\mathbf{v}$ onto $\mathbf{v}_1$ (denoted as $C_1\mathbf{v}_1$) to approximate $\mathbf{v}$, then the error is given by the component of $\mathbf{v}$ perpendicular to $\mathbf{v}_1$ (denoted as $\mathbf{v}_e$). Namely, $\mathbf{v} \approx C_1\mathbf{v}_1$, with error $\mathbf{v}_e$. Obviously, $\mathbf{v}_e$ is parallel to $\mathbf{v}_2$.

Usually we can use the vector projection of $\mathbf{v}$ onto the direction of $\mathbf{v}_1$ (i.e., $\hat{\mathbf{v}}_1$) to approximate $\mathbf{v}$, then the corresponding error is $\mathbf{v}_e$ (e standing for *error*). Let $C_1\mathbf{v}_1$ denote the component of vector $\mathbf{v}$ in the direction of $\mathbf{v}_1$, where C_1 is chosen to make the error vector as small as possible. Obviously the error takes the minimum when $\mathbf{v}_e$ is perpendicular to $\mathbf{v}_1$ and parallel to $\mathbf{v}_2$. It is seen,

- if $\mathbf{v}_e$ is zero and $C_1 = 1$, then $\mathbf{v}$ and $\mathbf{v}_1$ are identical in both magnitude and direction.
- if C_1 is zero, then the projection of $\mathbf{v}$ onto $\mathbf{v}_1$ is zero. They are said to be orthogonal. (4.4.13)

The above idea can be extended to the study of signals. Suppose we wish to approximate a signal $f(t)$

by another signal $f_1(t)$ over a certain interval $t_a < t < t_b$. Using the above idea we have

$$f(t) \approx C_1 f_1(t), \qquad t_a < t < t_b, \tag{4.4.14}$$

with an error

$$f_e(t) = f(t) - C_1 f_1(t). \tag{4.4.15}$$

If regarding $f(t)$ as the vector $\mathbf{v}$, $f_1(t)$ as $\mathbf{v}_1$ and $f_e(t)$ as $\mathbf{v}_e$, then according to Figure 4.9 we know $|\mathbf{v}_e|$ takes its minimum when

$$C_1\mathbf{v}_1 = (\mathbf{v}\cdot\hat{\mathbf{v}}_1)\,\hat{\mathbf{v}}_1 = \frac{\mathbf{v}\cdot\mathbf{v}_1}{|\mathbf{v}_1|^2}\mathbf{v}_1, \qquad \text{i.e.,} \qquad C_1 = \frac{\mathbf{v}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}, \tag{4.4.16}$$

with $|\mathbf{v}_1|^2 = \mathbf{v}_1\cdot\mathbf{v}_1$. In our present case, let us use integration to realize the inner products above, i.e.,

$$\mathbf{v}\cdot\mathbf{v}_1 = \int_{t_a}^{t_b} f(t)f_1(t)dt, \qquad \mathbf{v}_1\cdot\mathbf{v}_1 = \int_{t_a}^{t_b} f_1^2(t)dt. \tag{4.4.17}$$

Thus,

$$C_1 = \frac{\int_{t_a}^{t_b} f(t)f_1(t)dt}{\int_{t_a}^{t_b} f_1^2(t)dt}. \tag{4.4.18}$$

☞ **[Aside]:**

Another proof of eq.(4.4.18): Let us try to minimize the function $f_e(t)$. It should be noticed that, if we directly compute the average value of $f_e(t)$ over $[t_a, t_b]$, denoted as ϵ, in the following way:

$$\epsilon \stackrel{?}{=} \frac{1}{t_b - t_a}\int_{t_a}^{t_b} f_e(t)dt = \frac{1}{t_b - t_a}\int_{t_a}^{t_b} [f(t) - C_1 f_1(t)]\,dt, \tag{4.4.19}$$

then a critical difficulty will emerge that the positive and negative errors occurring at different instants would tend to cancel each other. This difficulty is avoidable if we choose to minimize the average squared-error $f_e^2(t)$ rather than the error itself, i.e.,

$$\epsilon = \frac{1}{t_b - t_a}\int_{t_a}^{t_b} f_e^2(t)dt = \frac{1}{t_b - t_a}\int_{t_a}^{t_b} [f(t) - C_1 f_1(t)]^2\,dt. \tag{4.4.20}$$

Then, in order to find the minimum value of ϵ let us differentiate eq.(4.4.20) with respect to C_1 and put the resulting expression equal to zero:

$$\frac{d}{dC_1}\left\{\frac{1}{t_b - t_a}\int_{t_a}^{t_b} [f(t) - C_1 f_1(t)]^2\,dt\right\} = 0. \tag{4.4.21}$$

Expanding the bracket and changing the order of integration and differentiation, we have

$$\frac{1}{t_b - t_a}\left[\int_{t_a}^{t_b} \frac{d}{dC_1} f^2(t)dt - 2\int_{t_a}^{t_b} f(t)f_1(t)dt + 2C_1\int_{t_a}^{t_b} f_1^2(t)dt\right] = 0. \tag{4.4.22}$$

The first integral vanishes since $f_1(t)$ is not a function of C_1. Thus (4.4.22) finally re-produces eq.(4.4.18):

$$C_1 = \frac{\int_{t_a}^{t_b} f(t)f_1(t)dt}{\int_{t_a}^{t_b} f_1^2(t)dt}. \tag{4.4.23}$$

Special cases, in agreement to the above discussions (4.4.13):

- When $\mathbf{v}\cdot\mathbf{v}_1 = \int_{t_a}^{t_b} f(t)f_1(t)dt = 0$, we have $C_1 = 0$, which means $f(t)$ and $f_1(t)$ are orthogonal to each other.
- When $\mathbf{v}\cdot\mathbf{v}_1 = \mathbf{v}_1\cdot\mathbf{v}_1$, we have $C_1 = 1$, that is, $\mathbf{v} = \mathbf{v}_1$, i.e., $f(t) = f_1(t)$.

✌

✎ ***Example* 4.4.** Suppose we wish to approximate the following square wave defined in the interval $0 < t < 2\pi/\omega$, as shown in figure 4.10:

$$f(t) = \begin{cases} 1, & \text{for} \quad 0 < t \le \frac{\pi}{\omega}, \\ -1, & \text{for} \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}. \end{cases} \qquad \omega \text{ — constant,} \tag{4.4.24}$$

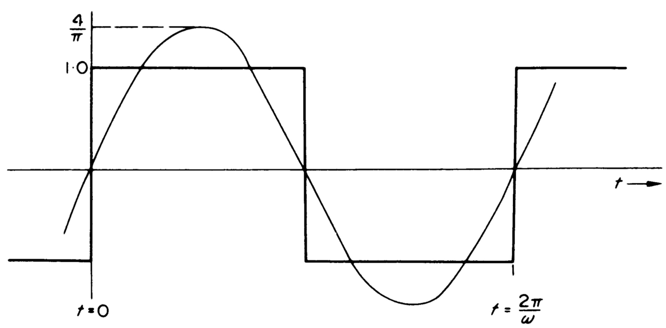


Figure 4.10: A square wave and its approximation by a sine wave of the same period.

We use the following sinusoidal wave to do the approximation, which has the same period as $f(t)$,

$$f_1(t) = \sin\omega t, \qquad 0 < t < \frac{2\pi}{\omega}. \tag{4.4.25}$$

The value of C_1, which minimizes the mean square error between the square wave and its approximation, is therefore given by

$$C_1 = \frac{\int_0^{\frac{2\pi}{\omega}} f(t)\sin\omega t\; dt}{\int_0^{\frac{2\pi}{\omega}} \sin^2\omega t\; dt} = \frac{\int_0^{\frac{\pi}{\omega}} \sin\omega t\; dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} (-\sin\omega t)\; dt}{\frac{\pi}{\omega}} = \frac{4}{\pi}. \tag{4.4.26}$$

This means

$$f_1(t) = \frac{4}{\pi} \sin \omega t \tag{4.4.27}$$

is the approximation of the square wave $f(t)$ which has the minimum mean square error. $f_1(t)$ is also shown in figure 4.10, superimposed upon the square wave $f(t)$.

$\Diamond$

4.4.2 Signal description by sets of orthogonal functions

If we wish to go a step further by improving the approximation, we need to appeal to other orthogonal functions to perform higher order approximation. Once again let us pay a revisit to vectors and consider the 3-dimensional space as shown in Figure 4.11

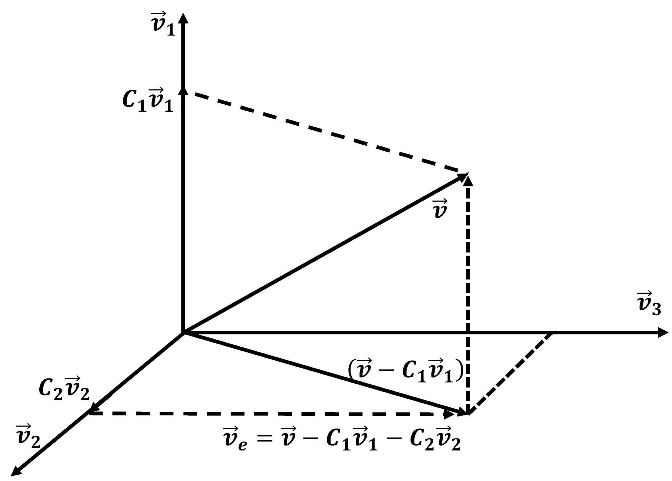


Figure 4.11: Computation of error in three dimensions. $\mathbf{v}$ is the original vector. In the space there are three orthogonal vectors, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. $C_1\mathbf{v}_1$ is the projection of $\mathbf{v}$ onto $\mathbf{v}_1$, $C_2\mathbf{v}_2$ is that onto $\mathbf{v}_2$, and the error is $\mathbf{v}_e = \mathbf{v} - C_1\mathbf{v}_1 - C_2\mathbf{v}_2$.

$\mathbf{v}$ is the original vector; we use $\mathbf{v}_1$ and $\mathbf{v}_2$ to approximate $\mathbf{v}$.

- The fist step is to use the projection of $\mathbf{v}$ onto $\mathbf{v}_1$ to approximate $\mathbf{v}$, $\mathbf{v} \approx C_1\mathbf{v}_1$, with error

  $$\mathbf{v}_e = \mathbf{v} - C_1\mathbf{v}_1. \tag{4.4.28}$$

  We take

  $$\mathbf{v}_e \perp \mathbf{v}_1, \qquad \text{i.e.,} \qquad C_1 = \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \tag{4.4.29}$$

  to guarantee the minimum evaluation of the error.

- The second step is to use the projection of $(\mathbf{v} - C_1\mathbf{v}_1)$ onto $\mathbf{v}_2$ to approximate $(\mathbf{v} - C_1\mathbf{v}_1)$, where $\mathbf{v}_2$ is a vector orthogonal to $\mathbf{v}_1$,

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0. \tag{4.4.30}$$

Let us use $C_2\mathbf{v}_2$ to denote this projection. Then, as usual, we take

$$\mathbf{v}_e = (\mathbf{v} - C_1\mathbf{v}_1) - C_2\mathbf{v}_2 \tag{4.4.31}$$

to play the role of error, and require

$$\mathbf{v}_e \perp \mathbf{v}_2, \qquad \text{i.e.,} \qquad C_2 = \frac{\mathbf{v} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \tag{4.4.32}$$

to guarantee the minimum evaluation of $\mathbf{v}_e$.

Obviously, $\mathbf{v}_e$ is in the direction of $\mathbf{v}_3$ that is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

Then, let us use $f(t)$, $f_1(t)$ and $f_2(t)$ to replace $\mathbf{v}$, $\mathbf{v}_1$ and $\mathbf{v}_2$. The inner products $\mathbf{v} \cdot \mathbf{v}_1$, $\mathbf{v} \cdot \mathbf{v}_2$ and $\mathbf{v}_1 \cdot \mathbf{v}_2$ are realized by, respectively,

$$\int_{t_a}^{t_b} f(t)f_1(t)dt, \qquad \int_{t_a}^{t_b} f(t)f_2(t)dt, \qquad \int_{t_a}^{t_b} f_1(t)f_2(t)dt, \tag{4.4.33}$$

where f_1 and f_2 are chosen to be orthogonal,

$$\int_{t_a}^{t_b} f_1(t)f_2(t)dt = 0. \tag{4.4.34}$$

Thus, we have the error

$$f_e = f(t) - C_1f_1(t) - C_2f_2(t), \tag{4.4.35}$$

where C_1 has been obtained as (4.4.18), and C_2 is given by

$$C_2 = \frac{\int_{t_a}^{t_b} f(t)f_2(t)dt}{\int_{t_a}^{t_b} f_2^2(t)dt}. \tag{4.4.36}$$

☞ **[Aside]:**

You may doubt if the expression for C_1, eq.(4.4.18), still holds in this 3-dimensional case. The answer is YES, as confirmed in the following.

For eq.(4.4.35), let us mimic eq.(4.4.20) to minimize the average squared-error

$$\epsilon = \overline{f_e^2} = \frac{1}{t_2 - t_1}\int_{t_a}^{t_b} \left[f(t) - C_1f_1(t) - C_2f_2(t)\right]^2 dt, \tag{4.4.37}$$

by doing the partial differentiation with respect to C_1 [*1]. By changing the order of differentiation and integration, we have the local extremum condition

$$\begin{aligned}\frac{\partial \epsilon}{\partial C_1} = \frac{1}{t_b - t_a} \quad & \left[\int_{t_a}^{t_b} \frac{\partial}{\partial C_1} f^2(t)dt + \int_{t_1}^{t_2} 2C_1 f_1^2(t)dt + \int_{t_a}^{t_b} \frac{\partial}{\partial C_1} C_2^2 f_2^2(t)dt \right. \\ & \left. - \int_{t_a}^{t_b} 2f(t)f_1(t)dt - \int_{t_a}^{t_b} \frac{\partial}{\partial C_1} 2f_(t)C_2 f_2(t)dt + \int_{t_a}^{t_b} 2f_1(t)C_2 f_2(t)dt \right] = 0.\end{aligned} \tag{4.4.38}$$

The first, third and fifth terms vanish trivially, since $f_1(t)$, $f_2(t)$ and $f_3(t)$ are not functions of C_1. The sixth term is also zero because $f_1(t)$ and $f_2(t)$ are set to be orthogonal in the interval t_a to t_b. Therefore, we arrive at the same result as before,

$$C_1 = \frac{\int_{t_a}^{t_b} f(t)f_1(t)dt}{\int_{t_a}^{t_b} f_1^2(t)dt}. \tag{4.4.39}$$

The above proof shows us a fact that incorporating an additional term in $f_2(t)$ does not require modifying the coefficient C_1, as long as $f_2(t)$ is orthogonal to $f_1(t)$ in the chosen time interval. Similarly it is not hard to prove that the value of C_2 does not change if the signal is approximated by $f_2(t)$ alone. This conclusion is crucial.

✌

The above result can be extended to more general cases, i.e., a larger set of orthogonal functions for the purpose of approximation. The use of orthogonal functions for signal description is analogous to the use of mutually perpendicular vectors for describing a vector in a higher dimensional space.

Consider a set $\{f_1(t), f_2(t), \cdots, f_n(t), \cdots\}$, where $f_1(t), f_2(t), \cdots, f_n(t), \cdots$ are infinitely many orthogonal functions over an interval $t_a < t < t_b$:

$$\int_{t_a}^{t_b} f_i(t)f_j(t)dt = 0, \qquad i \neq j, \quad f_i, f_j \neq 0. \tag{4.4.40}$$

Then a function/signal can be expressed as a series

$$f(t) = C_1 f_1(t) + C_2 f_2(t) + \cdots + C_n f_n(t) + \cdots, \tag{4.4.41}$$

where the expansion coefficients

$$C_i = \frac{\int_{t_a}^{t_b} f(t)f_i(t)dt}{\int_{t_a}^{t_b} f_i^2(t)dt}, \qquad i = 1, 2, \cdots, n, \cdots. \tag{4.4.42}$$

This result will have direct application in the coming Section 4.5.

[*1] We will have the chance to learn the knowledge of *Partial Differentiation* in this semester. See *Ping Yanru, Yao Hailou: Advanced Mathematics (Bilingual Course, II), Beijing University of Technology Press, 2015. Chapter 9.*

4.4.3 Typical examples of orthogonal functions

Sinusoidal functions

The first example for orthogonal functions is, as mentioned in Section 4.3, sinusoidal waves of different frequencies. Generally speaking, the following composite set of functions form an orthogonal set over an arbitrary interval $t_a < t < t_a + \frac{2\pi}{\omega}$ (— i.e., any interval equal to a period of the lowest frequency wave $\sin \omega t$ or $\cos \omega t$),

$$\{\sin n\omega t, \quad \cos n\omega t, \qquad n \in \mathbb{Z}, \;\; n \geq 1\}\,. \tag{4.4.43}$$

It is noted that when $n = 0$, $\sin n\omega t = 0$ and $\cos n\omega t = 1$, hence the complete orthogonal set may also take in the $n = 0$ cases and comprises

$$\{1, \cos \omega t, \cos 2\omega t, \cdots, \sin \omega t, \sin 2\omega t, \cdots\}\,. \tag{4.4.44}$$

Sinusoidal set of functions have wide applications in science and technology.

Orthogonal polynomial functions

Another example is the so-called Legendre polynomials, which is a set of mutually orthogonal polynomial functions over the interval $-1 < t < 1$. Its generating function is given by

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2 - 1)^n, \qquad n \in \mathbb{Z},\; n \geq 0. \tag{4.4.45}$$

Explicitly $\{P_n(x)\}$ is given by

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \left(\frac{3}{2}t^2 - \frac{1}{2}\right), \quad P_3(t) = \left(\frac{5}{2}t^3 - \frac{3}{2}t\right), \quad \cdots. \tag{4.4.46}$$

Similar examples of orthogonal polynomials include the Chebyshev polynomials, Walsh polynomials and so forth.

In a word, there are a number of orthogonal functions available for approximate description of signal waveforms. In our course we will focus on the sinusoidal function sets, eq.(4.4.43).

4.5 Fourier series

In 1822, Joseph Fourier published his seminal work in which he evolved the series bearing his name. Originally applied to the analysis of heat flow, the series has since been used in many branches of applied science, and constitutes one of the principal tools of signal analysis. A basic idea of this theory is that a complex periodic waveform may be analyzed with a number of harmonically-related sinusoidal and cosinusoidal waves serving as an orthogonal basis.
Let $f(t)$ be a periodic function/signal with period T. $f(t)$ can be expressed by a series, in the light of (4.4.41):

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\omega t + \sum_{n=1}^{\infty} B_n \sin n\omega t, \qquad \text{with } \omega = \frac{2\pi}{T}. \tag{4.5.47}$$

$f(t)$ is considered to be made up by the sum of

- a steady function A_0, and
- a number of sinusoidal and cosinusoidal waves of different frequencies.

  The lowest of these frequencies is ω (radian per second) and is called the *fundamental*; waves of this frequency have a period equal to that of the signal $f(t)$. Frequency 2ω is called the *second harmonic*, 3ω the *third harmonic*, and so on.

Certain restrictions, known as the Dirichlet condition, are placed upon $f(t)$ for its validity; fortunately, these conditions do not exclude the signal waveforms of practical interest.

4.5.1 Evaluation of coefficients

In the light of the general expression (4.4.42) for expansion coefficients, and writing (ωt) as x for convenience, the coefficients A_0, A_n and B_n have the following evaluations, respectively:

$$A_0 = \frac{\int_{-\pi}^{\pi} f(x) 1 dx}{\int_{-\pi}^{\pi} 1 dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \tag{4.5.48}$$

$$A_n = \frac{\int_{-\pi}^{\pi} f(x)\cos nx dx}{\int_{-\pi}^{\pi} \cos^2 nx dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \tag{4.5.49}$$

$$B_n = \frac{\int_{-\pi}^{\pi} f(x)\sin nx dx}{\int_{-\pi}^{\pi} \sin^2 nx dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx. \tag{4.5.50}$$

Discussion on parity: Many waveforms of practical interest are pure even or odd functions of time,

even function, $f(x) = f(-x)$; odd function, $f(x) = -f(-x)$.

- If $f(x)$ is an even function, then A_0, the integration of $f(x)$ over $[-\pi, \pi]$ does not vanish, where $[-\pi, \pi]$ is symmetric about $t = 0$. Meanwhile, $f(x) \sin nx$ is also odd since $\sin nx$ is odd. Thus the integrand for every B_n is odd, and its integration over $[-\pi, \pi]$ vanishes. Hence all the coefficients B_n's are zero; we are left with a series containing only the cosine functions, A_n's.
- On the contrary, if $f(x)$ is odd, then A_0 will vanish trivially; and by similar arguments we know the coefficients A's must be zero too. We are left with sine functions, B_n's, only.

☞ **[Aside]:**

In some cases, for a function $f(x)$ periodic on an interval $[-L, L]$ instead of $[-\pi, \pi]$, a simple change of variables can be used: $[-\pi, \pi] \longrightarrow [-L, L]$. Thus

$$x \longrightarrow \frac{\pi x}{L}, \qquad dx \longrightarrow \frac{\pi dx}{L}. \tag{4.5.51}$$

and

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \tag{4.5.52}$$

where

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx, \tag{4.5.53}$$

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos \frac{n\pi x}{L} dx, \tag{4.5.54}$$

$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \sin \frac{n\pi x}{L} dx. \tag{4.5.55}$$

If the function is instead defined on the interval $[0, 2L]$, the above equations simply become

$$A_0 = \frac{1}{2L} \int_{0}^{2L} f(x) dx, \tag{4.5.56}$$

$$A_n = \frac{1}{L} \int_{0}^{2L} f(x) \cdot \cos \frac{n\pi x}{L} dx, \tag{4.5.57}$$

$$B_n = \frac{1}{L} \int_{0}^{2L} f(x) \cdot \sin \frac{n\pi x}{L} dx. \tag{4.5.58}$$

In fact, for $f(x)$ periodic with period $2L$, any interval $(x_0, x_0 + 2L)$ can be used, with the choice being one of convenience or personal preference.

✌

✎ ***Example* 4.5.** Let us try to evaluate the coefficients of the sawtooth wave illustrated in figure 4.4. Noticing the $f(t)$ in this case is an odd function, we have

$$A_n = 0, \qquad B_n \neq 0, \qquad\qquad n = 1, 2, 3, \cdots .$$

The waveform is given by $f(t) = \frac{\omega t}{2}$ over the interval $-\frac{\pi}{\omega} < t < \frac{\pi}{\omega}$. Replacing ωt with x for convenience, and changing the limits to $x = \pm\pi$, we have

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x}{2}dx = \frac{1}{8\pi}\left[x^2\right]_{-\pi}^{\pi} = 0,$$

and

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2}\cdot \sin nx dx,$$

which is integrated by parts,

$$B_n = \frac{1}{2\pi}\left[\frac{\sin nx}{n^2} - \frac{x\cos nx}{n}\right]_{-\pi}^{\pi} = \frac{1}{\pi n^2}\left(\sin n\pi - n\pi\cos n\pi\right).$$

If n is an odd integer, $\sin n\pi = 0$ and $\cos n\pi = -1$, we have $B_n = 1/n$. If n is even, $\sin n\pi = 0$ and $\cos n\pi = 1$, we have $B_n = -1/n$. Thus

$$B_1 = 1, \quad B_2 = -\frac{1}{2}, \quad B_3 = \frac{1}{3}, \quad B_4 = -\frac{1}{4}, \cdots,$$

which exactly reproduces the Fourier expansion (4.3.11) of the sawtooth wave

$$f(t) = \sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \frac{1}{4}\sin 4\omega t + \cdots \tag{4.5.59}$$

[Remark]: A_0 is zero not because of the parity; it is due to the fact that the integral of $f(t)$ over a complete period is zero — in other words, it has a zero average.

◊

4.5.2 Choice of time origin and waveform power *(Optional)*

As shown in the last subsection, if a studied waveform is either an even or odd function, the calculation of its Fourier coefficients A_0, A_n and B_n becomes much simpler. To make this happen, a wise choice is to (re-)arrange the time origin, i.e., to use translation of the integration variable t to produce even/odd functions.
For example, Figure 4.12 shows three versions of a square wave which differ only in their time origin.

- Wave (b) is an odd function, anti-symmetric about $t = 0$. It is seen in eq.(4.4.27) that its fundamental component is $\frac{4}{\pi}\sin\omega t$.
- Wave (a) is identical except that it is an even function with a fundamental equal to $\frac{4}{\pi}\cos\omega t$.

  This shift of time origin therefore merely has the effect of converting a Fourier series containing only sine terms into that containing only cosine terms. Its amplitude of a component is, as expect, unaltered at any frequency.
- Wave (c) is however more complicated since the square wave is neither even nor odd; expectedly it include both sine and cosine terms in its Fourier series.

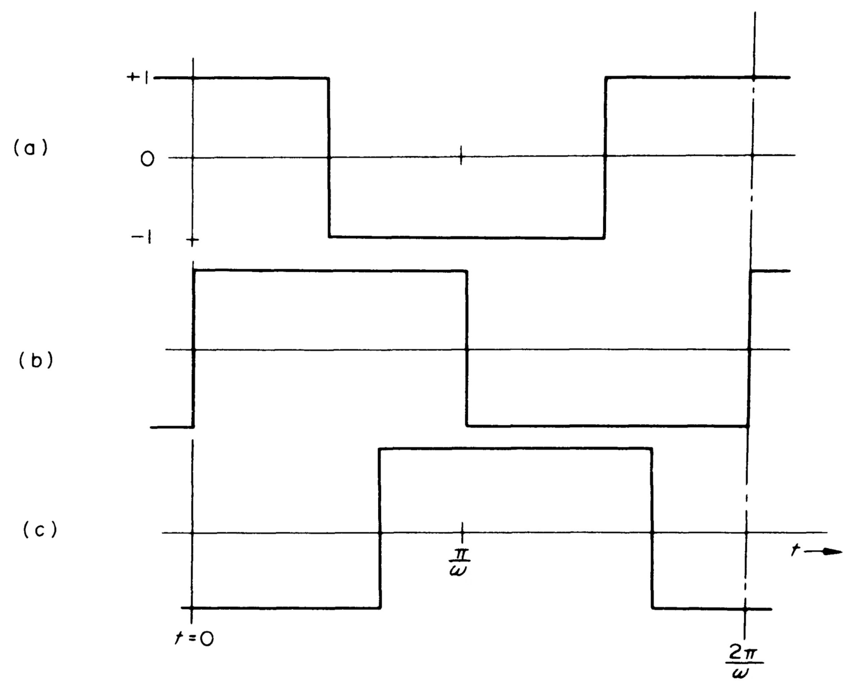


Figure 4.12: Three square waves, which are essentially identical, but apart from a time-shift.

The clue to the relationship between the values of the various coefficients in case (c) and those in (a) and (b) lies in the average power of the waveform, a concept familiar to electrical engineers. Suppose we find when we analyze the waveform of Figure 4.12(c) that there are fundamental components

$$A_1 \cos \omega t \qquad \text{and} \qquad B_1 \sin \omega t$$

If the component $A_1 \cos \omega_1 t$ represents a voltage applied to a resistor of value 1 Ohm, then the average power P dissipated by it in the resistor (recall that Power = Voltage^2/Resistance) over one complete period will be:

$$P = \frac{1}{2\pi} \int_{-\pi}^{\pi} (A_1 \cos x)^2 dx = \frac{A_1^2}{2} \tag{4.5.60}$$

where $x = \omega_1 t$. In other words the mean power dissipated is equal to the average squared value of the voltage waveform. Similarly, the average squared value of the wave $B_1 \sin \omega_1 t$ over one period is $B_1^2/2$. The total power represented by the two fundamental components together is thus $\frac{1}{2}\left(A_1^2 + B_1^2\right)$. It is clear however that this value will be the same for all three examples of the square wave of Figure 4.12, since the average power represented by a waveform is not altered by a mere shift in time origin. Since for waveform (a) we have already found that $A_1 = 4/\pi$ and $B_1 = 0$, and for waveform (b) $B_1 = 4/\pi$ and $A_1 = 0$, we conclude that for any other waveform such as (c)

$$\left(\frac{A_1^2 + B_1^2}{2}\right) = \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \tag{4.5.61}$$

thus

$$\left(A_1^2 + B_1^2\right) = \left(\frac{4}{\pi}\right)^2 \tag{4.5.62}$$

Hence as the time origin of a waveform is shifted, the various sine and cosine coefficients of its Fourier series will change, but the sum of the squares of any two coefficients A_n and B_n will remain constant.

The above ideas lead naturally to an alternative trigonometric form for the Fourier series. If the two fundamental components of a waveform are $A_1 \cos \omega_1 t$ and $B_1 \sin \omega_1 t$ their sum may be expressed in an alternative form using trigonometric identities

$$\begin{aligned} A_1 \cos \omega t + B_1 \sin \omega_1 t &= \sqrt{A_1^2 + B_1^2} \cos\left(\omega t - \tan^{-1} \frac{B_1}{A_1}\right) \\ &= \sqrt{A_1^2 + B_1^2} \sin\left(\omega t + \tan^{-1} \frac{A_1}{B_1}\right). \end{aligned} \tag{4.5.63}$$

Thus the sine and cosine components at a particular frequency are expressed as a single cosine or sine wave together with a phase shift. This equivalence is illustrated in Figure 4.13. If this procedure is applied to all harmonic components of the Fourier series, we get the alternative forms

$$f(t) = A_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega t - \phi_n), \qquad \text{or} \qquad f(t) = A_0 + \sum_{n=1}^{\infty} C_n \sin(n\omega t + \theta_n). \tag{4.5.64}$$

where

$$C_n = \sqrt{A_n^2 + B_n^2}, \qquad \phi_n = \tan^{-1} \frac{B_n}{A_n}, \qquad \theta_n = \tan^{-1} \frac{A_n}{B_n}. \tag{4.5.65}$$

Finally, we note that since the mean power represented by any component wave is

$$\frac{A_n^2 + B_n^2}{2} = \frac{C_n^2}{2} \tag{4.5.66}$$

and the power represented by the term A_0 is simply A_0^2, the total average waveform power is equal to

$$P = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 \tag{4.5.67}$$

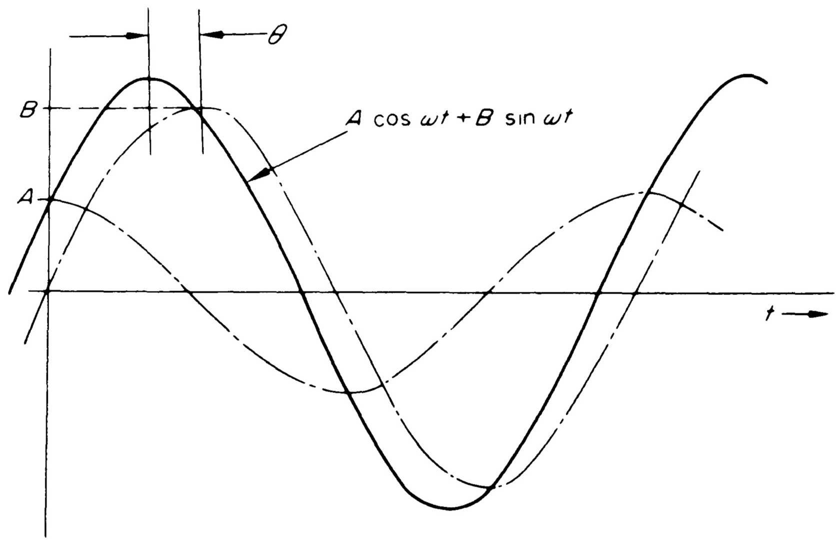


Figure 4.13: The addition of a sine and a cosine function to give a wave of the same frequency with a phase angle θ.

But P may also be expressed as the average value over one period of $[f(t)]^2$, using again the convention that $f(t)$ is considered to represent a voltage waveform applied across a 1 Ohm resistor. Hence

$$P = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 = \frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2\, dt. \tag{4.5.68}$$

This result is a version of a more general one known as Parseval's theorem, and shows that the total waveform power is equal to the sum of the powers represented by its individual Fourier components. It is, however, important to note that this is only true because the various component waves are drawn from an orthogonal set. This may be shown by considering a wave $f(t)$ which contains just the two components

$$A_n \cos n\omega t + B_m \sin m\omega t. \tag{4.5.69}$$

The average total waveform power is equal to the average value taken over one period of $(A_n \cos n\omega t + B_m \sin m\omega t)^2 = (A_n \cos n\omega t)^2 + (B_m \sin m\omega t)^2 + 2A_nB_m \cos n\omega t \cdot \cos n\omega t$. The average value of the last term(over one period) is zero for any values of m and n, because $\cos n\omega t$ and $\sin m\omega t$ are orthogonal functions. Hence the total average waveform power is

$$\frac{A_n^2 + B_m^2}{2}. \tag{4.5.70}$$

and is equal to the sum of the powers in the two individual components. A similar result is obtained for more complex waveforms having more Fourier components, since they are all members of an orthogonal set.

Some general comments

It is often possible to anticipate the main characteristics of a periodic waveform just from a visual inspection. For example, a signal which exhibits sudden changes or discontinuities such as that of Figure 4.4 must be expected to be rich in the higher-order harmonics, because it is only possible to build up such a waveform using component waves which are themselves changing rapidly(that is, high frequency waves).

Chapter 5 Fourier series and transformations (II) — Aperiodic functions

In eq.(4.3.11) of the previous chapter it was shown how a periodic signal may be expressed as the sum of a set of sinusoidal waves which are harmonically related (*fundamental, second harmonic, third harmonic, etc.*). The spectrum of such a signal consists of a number of discrete frequencies known as a *line* spectrum. Although the analysis of periodic signals gives results of much practical interest, the great majority of signals are more general beyond that type. Firstly, there are signals/functions which may not generally be assumed to exist for all time; e.g., the Heaviside step function,

$$H(x) = \begin{cases} 0, & \text{when } x < 0, \\ 1, & \text{when } x \geq 0. \end{cases} \tag{5.0.1}$$

Secondly, there is an important type of signal waveforms which are simply not repetitive in nature and cannot therefore be represented by Fourier series containing a number of harmonically-related discrete frequencies. Fortunately, however, it is possible to generalize the knowledge of discrete Fouries series to continuous frequency spectra to analyze these signals. In the following the exponential form is particularly helpful for the derivation of frequency spectra of aperiodic signals.

5.1 Exponential form of Fourier series

5.1.1 Discrete exponential form of Fourier series

The Fourier expansion of a function $f(t)$ with respect to discrete sine and cosine functions is given by eq.(4.5.47),

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + \sum_{n=1}^{\infty} B_n \sin nx, \qquad \text{with } x = \omega t,\ \omega = \frac{2\pi}{T}, \tag{5.1.2}$$

where the coeffeicients A_0, A_n and B_n are given by eqs.(4.5.48), (4.5.49) and (4.5.50).

Given the Euler formula (3.1.1), we have the exponential expressions for the sine and cosine functions, eq.(3.1.6):

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right). \tag{5.1.3}$$

Substituting (5.1.3) into (5.1.2), we have

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty} A_n \frac{e^{inx} + e^{-inx}}{2} - \sum_{n=1}^{\infty} iB_n \frac{e^{inx} - e^{-inx}}{2} \\ &= A_0 + \sum_{n=1}^{\infty} \frac{1}{2}\left(A_n - iB_n\right) e^{inx} + \sum_{n=1}^{\infty} \frac{1}{2}\left[A_{-(-n)} + iB_{-(-n)}\right] e^{i(-n)x} \qquad \text{(letting } m = -n\text{)} \\ &= A_0 + \sum_{n=1}^{\infty} \frac{1}{2}\left(A_n - iB_n\right) e^{inx} + \sum_{m=-\infty}^{-1} \frac{1}{2}\left(A_{-m} + iB_{-m}\right) e^{imx}, \qquad x = \omega t. \end{aligned} \tag{5.1.4}$$

Introducing new coefficients

$$a_m = \begin{cases} \frac{1}{2}\left(A_{-m} + iB_{-m}\right), & \text{when } m \le -1, \\ A_0, & \text{when } m = 0, \\ \frac{1}{2}\left(A_m - iB_m\right), & \text{when } m \ge 1, \end{cases} \qquad m \in \mathbb{Z}, \tag{5.1.5}$$

the function $f(x)$ can be re-expressed as

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty} a_m e^{imx} \\ &= \cdots + a_{-2}e^{-i2x} + a_{-1}e^{-ix} + a_0 + a_1 e^{ix} + a_2 e^{i2x} + \cdots, \quad m \in \mathbb{Z}. \end{aligned} \tag{5.1.6}$$

It is addressed that, although the introduction of complex coefficients is somehow not easy to accept at first, one may recollect the memory

- the real part of a pair of coefficients denotes the magnitude of the cosine wave of the relevant frequency, and
- the imaginary part denotes the magnitude of the sine wave.

If a particular pair of coefficients a_n and a_{-n} are real, the component at the frequency $n\omega$ is simply a cosine; if a_n and a_{-n} are purely imaginary, the component is just a sine; and if, as is the general case, a_n and a_{-n} are complex, both a cosine and a sine term are present.

☞ **[Aside]:**

The use of the exponential form of the Fourier series gives rise to a further notion which is often found difficult, that of *negative frequency*. Of course, a cosine $A\cos\omega t$ is a wave of a single frequency ω radians/second, and may be represented by a single line of height A in a spectral diagram. If however we are using the exponential form of the Fourier series and are discussing a waveform in terms of its exponential components, we use the identity

$$A\cos\omega t = \frac{A}{2}(e^{i\omega t} + e^{-i\omega t}) \tag{5.1.7}$$

Plotting the exponential components on a spectral diagram, we now consider the term $Ae^{i\omega t}/2$ to be represented by a line of height $A/2$ at a frequency ω, and the term $Ae^{-i\omega t}/2 = Ae^{\mathrm{j}(-\omega)t}/2$ to be represented by a line of height $A/2$ at a frequency $-\omega$. Thus our frequency scale is now formally extended to include negative as well as positive frequencies, and a cosine component in a signal waveform gives rise to two spectral lines. Similarly, a sine component gives rise to two equal but opposite imaginary exponential components, which cannot of course be plotted on the same spectral diagram as the real exponential components representing the cosines. So a complete spectral description of a signal waveform will normally involve two separate diagrams, one representing real exponential terms (cosines) and the other representing imaginary terms (sines), as shown in Figure 5.1. It is therefore important to remember that the introduction of *negative* frequencies implies that sines and cosines are being represented in exponential form.

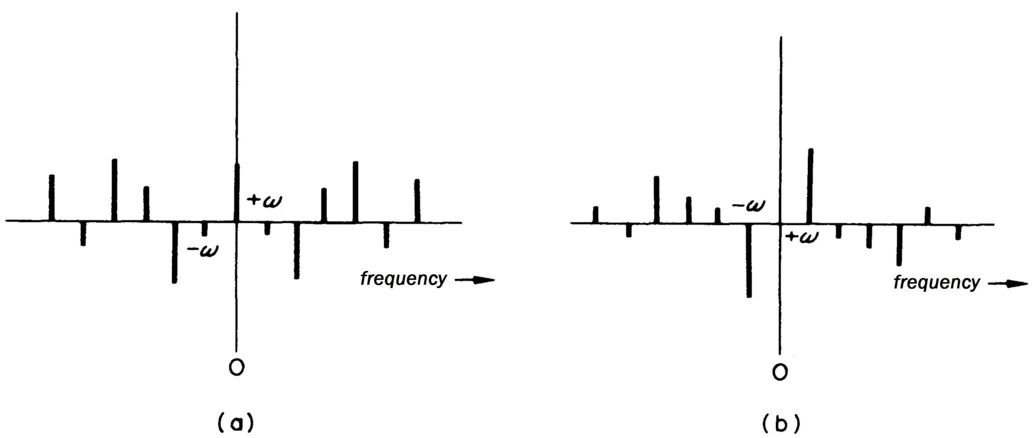


Figure 5.1: Exponential representation of a Fourier Series: (a) Real parts of the exponential coefficients representing cosine components, and (b) imaginary parts representing sine components.

5.1.2 Coefficients expressed in exponential form

In eq.(5.1.5) we obtain the expression for a_m for the cases of $m > 0$, $m < 0$ and $m = 0$, where $m \in \mathbb{Z}$. Given that A_0, A_n and B_n have their definitions in (4.5.48), (4.5.49) and (4.5.50), respectively, we have the following computations:

- When $m \leq -1$,

$$\begin{aligned} a_m &= \frac{1}{2}\left(A_{-m} + iB_{-m}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(-mx)dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)i\sin(-mx)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-imx}dx. \end{aligned} \tag{5.1.8}$$

- When $m = 0$,

$$a_m = A_0 = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)e^{i0x}dx. \tag{5.1.9}$$

- When $m \geq 1$,

$$\begin{aligned} a_m &= \frac{1}{2}\left(A_m - iB_m\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)i\sin(mx)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx. \end{aligned} \tag{5.1.10}$$

Summarizing (5.1.8), (5.1.9) and (5.1.10) we achieve a universal expression

$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx, \qquad m \in \mathbb{Z},\ x = \omega t. \tag{5.1.11}$$

☞ **[Aside]:**

From discrete pulse spectra to continuous sinusoidal wave acting as the envelope of the pulse spectra:
We now turn our attention to the analysis of the recurrent pulse waveform of Figure 5.1.2(a), This wave is important for two main reasons: firstly, it is of great practical interest because similar waveforms occur widely in such devices as digital computers and communication systems; and secondly it is of analytic interest because it provides a good starting point for a discussion of the Fourier transform.

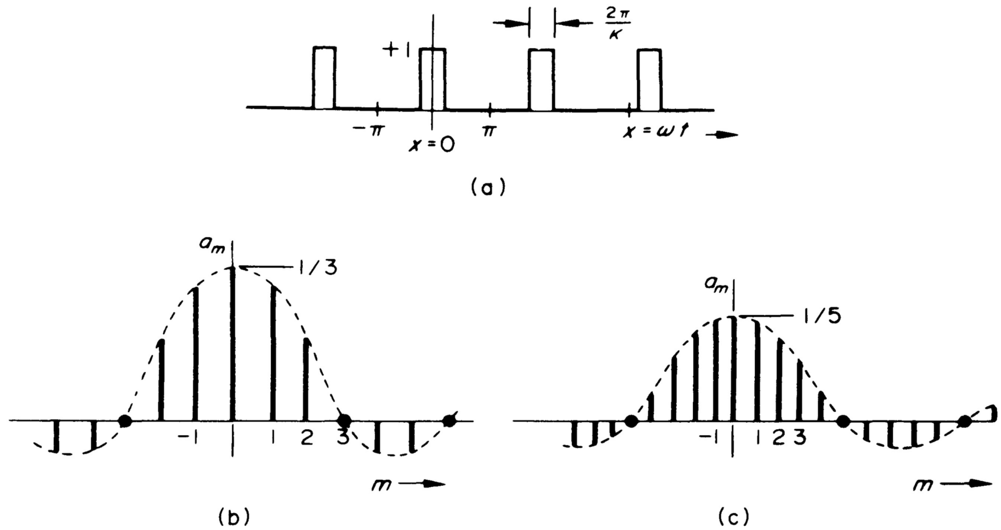


Figure 5.2: (a) A repetitive pulse waveform, and its real exponential Fourier coefficients for (b) K = 3, and (c) K = 5

For convenience we will assume (as in the diagram) that the period of the waveform is K times as large as the pulse duration. If we consider the period of the waveform between $x = \pm\pi$, it is clear that a finite contribution to the integral occurs only in the interval $x = \pm\pi/K$, where the pulse height is unity. Hence

$$a_m = \frac{1}{2\pi}\int_{-\pi/K}^{\pi/K} e^{-imx}dx, \qquad x = \omega t. \tag{5.1.12}$$

If $m = 0$, $a_0 = \frac{1}{2\pi}\left(\frac{\pi}{K} + \frac{\pi}{K}\right) = \frac{1}{K}$; if $m \neq 0$,

$$a_m = \frac{1}{-im2\pi}\left[e^{-imx}\right]_{-\pi/K}^{\pi/K} = \frac{1}{m\pi}\frac{e^{im\pi/K} - e^{-im\pi/K}}{2i} = \frac{1}{K}\frac{\sin(m\pi/K)}{m\pi/K} \tag{5.1.13}$$

Conversely, the recurrent pulse may be synthesised by summing components as follows

$$f(x) = \sum_{m=-\infty}^{\infty}\frac{1}{K}\frac{\sin(m\pi/K)}{m\pi/K}\cdot e^{im\pi/K} \tag{5.1.14}$$

Figure (b) and (c) illustrate this result for $K = 3$ and $K = 5$. As K increases, the harmonic terms become closer spaced under the $(\sin x/x)$ *envelope* and they reduce correspondingly in absolute size; the actual frequency represented by any line depends on the absolute period of the time function. If K becomes very large, the pulse duration is very small in comparison with the waveform period. The spectrum consists of a correspondingly large number of spectral lines, very closely bunched and of vanishingly small amplitude. In the limit the lines become so close that we call the spectrum *continuous*; then we are led with little difficulty to the concept of Fourier transform.

✌

5.2 Fourier transforms

5.2.1 Derivation of Fourier integral: Treatment from discrete to continuous

In the last Section 5.1 it is shown that a discrete Fourier series is suitable for describing a periodic signal (which repeats itself periodically). In this section we will show that, letting the *period* T go to infinity, i.e., letting $\omega = \frac{2\pi}{T}$ go to 0, a discrete Fourier series is turned into a continuous Fourier integral. Since $T \to \infty$ implies an aperiodic signal, the Fourier integral is suitable for describing a generic function no matter it is periodic or aperiodic. *The reader is strongly suggested to pay close attention to the technique in this section of turning a discrete summation into a continuous integral, which is very useful and popular in practice in science and technology.*

Let us start from the exponential forms of the Fourier series, eq.(5.1.6), and the coefficients, (5.1.11):

$$f(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx}, \tag{5.2.15}$$

$$\text{with} \quad a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx, \qquad m \in \mathbb{Z},\ \ x = \omega t. \tag{5.2.16}$$

Keeping in mind $\omega = \frac{2\pi}{T}$, we have

$$f(t) = \sum_{m=-\infty}^{\infty} a_m e^{im\omega t}, \tag{5.2.17}$$

$$\text{with} \quad a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-im\omega t}d(\omega t) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-im\omega t}dt. \tag{5.2.18}$$

Defining a function

$$G_m = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-im\omega t}dt, \tag{5.2.19}$$

we have

$$a_m = \frac{G_m}{T}, \tag{5.2.20}$$

$$f(t) = \sum_{m=-\infty}^{\infty} G_m \frac{e^{im\omega t}}{T} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} G_m e^{im\omega t}\omega. \tag{5.2.21}$$

Now we are at stage of turning the discrete series into a continuous integral.
Consider the case that $T \gg 1$, i.e., $\omega \ll 1$, then one can do the following replacements:

$$\omega \Longrightarrow \Delta\omega, \qquad m\omega \Longrightarrow \omega. \tag{5.2.22}$$

Under the limitation behavior $\Delta\omega \to 0$, the $\Delta\omega$ becomes a differential $d\omega$; and G_m, that depends on the discrete counting of m, becomes a function of the continuous variable ω : $G(\omega)$. Then eqs.(5.2.21) and (5.2.19) become the integrals

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega, \tag{5.2.23}$$

$$\text{with} \qquad G(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \tag{5.2.24}$$

where the upper and lower bounds of $G(\omega)$ has the limits: $\int_{-\frac{T}{2}}^{\frac{T}{2}} \to \int_{-\infty}^{\infty}$, when $T \to \infty$.

[Remarks]:

- Eq.(5.2.23) is the desired Fourier transform (i.e., Fourier integral) of the function $f(t)$, with $G(\omega)$ of (5.2.24) called the kernel of the transform.

  In literature people use $G(\omega) \equiv \mathscr{F}[f(t)]$ to denote Fourier transform.

- $G(\omega)$ and $f(t)$ are in a dual relation; meanwhile, the frequency ω (in the unit Hz $= \frac{1}{\text{second}}$) and the time t (in the unit of second) are a pair of dual variables, no forgetting ωt is dimensionless. (Recall Figure 4.6.)

- It is addressed that in the derivation of the formulae the period T has the limitation behavior $T \to \infty$, which means the Fourier integration method is suitable to describe a general function, no matter it is periodic or aperiodic.

- Are eqs.(5.2.23) and (5.2.24) correct? Let us have a quick double check.

  Substituting (5.2.24) into (5.2.23), we have

$$\begin{aligned} RHS &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt'\right]e^{i\omega t}d\omega \qquad \text{(swapping the order of the integrals)} \\ &= \int_{-\infty}^{\infty} dt' f(t')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}d\omega\right] \\ &= \int_{-\infty}^{\infty} dt' f(t')\delta(t-t') = f(t) = LHS, \end{aligned} \tag{5.2.25}$$

  where $\delta(x)$ is the so-called Dirac δ-function defined as

$$\delta(x) = \begin{cases} 0, & \text{when } x \neq 0, \\ \infty, & \text{when } x = 0, \end{cases} \tag{5.2.26}$$

$$\text{satisfying} \qquad \int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{5.2.27}$$

δ-function has a property

$$\int_{-\infty}^{\infty} f(x')\delta(x-x')dx' = f(x). \tag{5.2.28}$$

The Fourier transform of δ-function reads

$$\delta(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\omega. \tag{5.2.29}$$

- Usually the following variable replacements apply:

$$t \to x, \qquad \omega \to k, \tag{5.2.30}$$

such that a Fourier transform $G(k) \equiv \mathscr{F}[f(x)]$ is obtained,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)\cdot e^{ikx} dk, && (5.2.31)\\ G(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx. && (5.2.32) \end{aligned}$$

In this picture x carries the meaning of displacement (in the unit of meter), and k the meaning of momentum (in the unit of $\frac{\text{meter}}{\text{second}}$).[*1] This is used to describe propagation of transverse waves with given speed and shape.

[*1] Notice: Here the natural unit system applies, namely, $c = \hbar = 1$, where c as a velocity is in the unit $\frac{\text{meter}}{\text{second}}$, and $\hbar$ as an angular momentum in the unit $\frac{\text{meter}^2}{\text{second}}$.

5.2.2 Properties *(Optional)*

In this section some basic theorems/properties of Fourier transform are collected.
In the following the symbol $\mathscr{F}$ denotes an operation of Fourier transform. The constants $\alpha_1, \alpha_2 \in \mathbb{R}$, and the functions $f_1(x), f_2(x) \in \mathbb{R}$ with $x \in \mathbb{R}$.

***Theorem* 5.1.**

$$\mathscr{F}[\alpha_1 f_1(x) + \alpha_2 f_2(x)] = \alpha_1 \mathscr{F}[f_1(x)] + \alpha_2 \mathscr{F}[f_2(x)]. \tag{5.2.33}$$

Proof:

$$\begin{aligned}
\mathscr{F}[\alpha_1 f_1(x) + \alpha_2 f_2(x)] &= \int_{-\infty}^{\infty} [\alpha_1 f_1(x) + \alpha_2 f_2(x)] e^{-ikx} dx \\
&= \alpha_1 \int_{-\infty}^{\infty} f_1(x) e^{-ikx} dx + \alpha_2 \int_{-\infty}^{\infty} f_2(x) e^{-ikx} dx \\
&= \alpha_1 \mathscr{F}[f_1(x)] + \alpha_2 \mathscr{F}[f_2(x)].
\end{aligned}$$

□

***Theorem* 5.2.**

$$\mathscr{F}[f(x - x_0)] = e^{-ikx_0} \mathscr{F}[f(x)]. \tag{5.2.34}$$

Proof:

$$\begin{aligned}
\mathscr{F}[f(x - x_0)] &= \int_{-\infty}^{\infty} f(x - x_0) e^{-ikx} dx = \int_{-\infty}^{\infty} f(x - x_0) e^{-ik(x - x_0)} e^{-ikx_0} dx \\
&= e^{-ikx_0} \int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi = e^{-ikx_0} \mathscr{F}[f(x)],
\end{aligned}$$

where $\xi = x - x_0$. □

***Theorem* 5.3.**

$$\mathscr{F}[f(x)e^{ikx_0}] = G(k - k_0). \tag{5.2.35}$$

Proof:

$$\mathscr{F}[f(x)e^{ikx_0}] = \int_{-\infty}^{\infty} f(x)e^{-i(k-k_0)x}dx = G(k - k_0).$$

□

***Theorem* 5.4.**

$$\mathscr{F}[f(ax)] = \frac{1}{\mid a \mid} G\left(\frac{k}{a}\right), \tag{5.2.36}$$

where $a \neq 0$.

Proof: Here we use $x' = ax$ instead of x, when $a > 0$.

$$\mathscr{F}[f(ax)] = \int_{-\infty}^{\infty} f(ax)e^{-ikx}dx = \frac{1}{a}\int_{-\infty}^{\infty} f(x')e^{-i\frac{k}{a}x'}dx' = \frac{1}{\mid a \mid} G\left(\frac{k}{a}\right),$$

when $a < 0$

$$\begin{aligned}\mathscr{F}[f(ax)] &= \int_{-\infty}^{\infty} f(ax)e^{-ikx}dx = \frac{1}{a}\int_{\infty}^{-\infty} f(x')e^{-i\frac{k}{a}x'}dx' \\ &= \frac{1}{-a}\int_{-\infty}^{\infty} f(x')e^{-i\frac{k}{a}x'}dx' = \frac{1}{\mid a \mid} G\left(\frac{k}{a}\right).\end{aligned}$$

□

***Theorem* 5.5.**

$$\mathscr{F}[f'(x)] = ik\mathscr{F}[f(x)], \tag{5.2.37}$$

$$\mathscr{F}[f^{(n)}(x)] = (ik)^n\mathscr{F}[f(x)], \tag{5.2.38}$$

if $\mid x \mid \to \infty$ then $f(x) \to 0$.

Proof:

$$\mathscr{F}[f'(x)] = \int_{-\infty}^{\infty} f'(x)e^{-ikx}dx = f(x)e^{-ikx}\Big|_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = ik\mathscr{F}[f(x)].$$

From $\mathscr{F}[f''(x)] = ik\mathscr{F}[f'(x)] = (ik)^2\mathscr{F}[f(x)]$, we have

$$\mathscr{F}[f^{(n)}(x)] = (ik)^n\mathscr{F}[f(x)].$$

□

***Theorem* 5.6.**

$$\mathscr{F}[f_1(x) * f_2(x)] = \mathscr{F}[f_1(x)]\mathscr{F}[f_2(x)], \tag{5.2.39}$$

where

$$f_1(x) * f_2(x) \equiv \int_{-\infty}^{\infty} f_1(x-\xi)f_2(\xi)d\xi. \tag{5.2.40}$$

The function above is called the convolution of $f_1(x)$ and $f_2(x)$.

Proof:

$$\begin{aligned}
\mathscr{F}[f_1(x) * f_2(x)] &= \int_{-\infty}^{\infty} dx e^{-ikx} \int_{-\infty}^{\infty} f_1(x-\xi)f_2(\xi)d\xi \\
&= \int_{-\infty}^{\infty} d\xi f_2(\xi) \int_{-\infty}^{\infty} f_1(x-\xi)e^{-ikx}dx \\
&= \int_{-\infty}^{\infty} d\xi f_2(\xi)e^{-ik\xi} \int_{-\infty}^{\infty} f_1(x-\xi)e^{-ik(x-\xi)}dx \\
&= \mathscr{F}[f_1(x)]\mathscr{F}[f_2(x)].
\end{aligned}$$

□

[Summary of formulae]

$$\begin{aligned}
\mathscr{F}[\alpha_1 f_1(x) + \alpha_2 f_2(x)] &= \alpha_1\mathscr{F}[f_1(x)] + \alpha_2\mathscr{F}[f_2(x)] \\
\mathscr{F}[f(x-x_0)] &= e^{-ikx_0}\mathscr{F}[f(x)] \\
\mathscr{F}[f(x)e^{ikx_0}] &= G(k-k_0) \\
\mathscr{F}[f(ax)] &= \frac{1}{\mid a \mid}G\left(\frac{k}{a}\right) \\
\mathscr{F}[f'(x)] &= ik\mathscr{F}[f(x)] \\
\mathscr{F}[f^{(n)}(x)] &= (ik)^n\mathscr{F}[f(x)] \\
\mathscr{F}[f_1(x) * f_2(x)] &= \mathscr{F}[f_1(x)]\mathscr{F}[f_2(x)]
\end{aligned}$$

The asterisk($*$) denotes the convolution of $f_1(x)$ and $f_2(x)$.